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NON-LINEAR ANALYSIS OF THREE-PINNED CIRCULAR ARCHES

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Because a three-pinned circular arch is statically determinate, when it is subjected to a uniform radial load q, linear in-plane analysis has shown that the uniform load will produce quite simple internal actions: a uniform axial compressive force $N = qR$ and zero-bending moment, where R is the radius of the arch. This is consistent with equations in textbooks for structural mechanics. However, the non-linear behavior and buckling of three-pinned arches are very different from their linear counterparts. The uniform radial load can produce significant bending moments in the three-pinned arches, and the value of the uniform axial compressive force in the three-pinned arches is greater than qR. In addition, it is also shown in this paper that the solutions for the in-plane elastic buckling load of three-pinned arches available in the open literature cannot predict their in-plane buckling loads correctly.

Keywords: Structural mechanics, Arch analysis, Buckling, Serviceability, Limit state.

1 INTRODUCTION

Because three-pinned arches are statically determinate, it is commonly considered that linear analysis is sufficiently accurate for predicting their in-plane elastic behavior and buckling. However, it has been shown (Bradford et al. 2002, Pi et al. 2002) that linear analysis is not adequate for predicting the in-plane structural response of two-pinned shallow arches. Hence, linear analysis may also not be adequate for predicting the in-plane structural response of three-pinned shallow arches. In addition, because of the free rotation of a pinned crown, deformations of three-pinned arches are much larger than those of the two-pinned arches, and the structural response becomes non-linear earlier than that of two-pinned arches. Hence, a three-pinned arch, whether it is shallow or deep, may be quite susceptible to non-linear behavior and to subsequent buckling.

This paper presents a non-linear analysis for the in-plane elastic structural behavior and buckling of three-pinned circular arches under a uniform radial load q (Figure 1). It compares the "conventional" linear analysis with non-linear analysis to demonstrate that non-linear analysis is required for correct predictions of in-plane structural behavior and buckling of three-pinned arches.

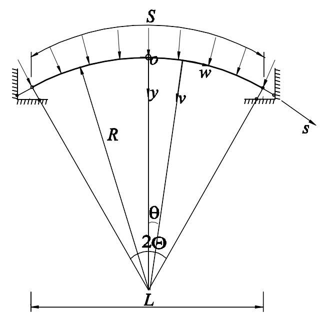

Figure 1. Three-pinned circular arch.

2 LINEAR ANALYSIS

When a three-pinned steel circular arch is subjected to a uniform radial load, the differential equations of equilibrium for linear analysis can be written as (Pi et al. 2013):

$$EI(\tilde{v}^{iv}+\tilde{w}''') - AER^2(\tilde{w}'-\tilde{v}) - qR^3 = 0 \text{, and} \quad EI(\tilde{v}'''+\tilde{w}'') - AER^2(\tilde{w}''-\tilde{v}') = 0 \tag{1}$$

in the radial and axial directions, where $\tilde{v} = v/R$ and $\tilde{w} = w/R$, v and w are the radial and axial displacements, and $(\,)' = \mathrm{d}(\,)/\mathrm{d}\theta$, θ is the angular coordinate (Figure 1). The static boundary conditions can be obtained as:

$$\tilde{v}'' + \tilde{w}' = 0 \text{ and } \tilde{v}''' + \tilde{w}'' = 0 \text{ at } \theta = 0 \tag{2}$$

The kinematic essential boundary conditions are:

$$v = 0 \text{ at } \theta = \pm\Theta, \text{ and } w = 0 \text{ at } \theta = 0 \text{ and } \theta = \pm\Theta. \tag{3}$$

Solving two equations of Eq. (1) and considering the boundary conditions given by Eqs. (2) and (3) leads to the solutions for the radial and axial displacements as:

$$\tilde{v} = \frac{qR}{AE}\left[1+\cos\theta - \mathrm{H}(\theta)\sin\theta\cot\frac{\Theta}{2}\right] \tag{4}$$

and:

$$\tilde{w} = \frac{qR}{AE}\left[\sin\theta - \mathrm{H}(\theta)(\cos\theta - 1)\cot\frac{\Theta}{2}\right] \tag{5}$$

where $\mathrm{H}(\theta)$ is a step function such that $\mathrm{H}(\theta) = 1$ when $\theta > 0$, and $\mathrm{H}(\theta) = -1$ when $\theta < 0$.

The axial compressive force and bending moment of the three-pinned arch can be obtained by substituting Eqs. (4) and (5) as:

$$N = aE(\tilde{w}' - \tilde{v}) = qR \quad \text{and} \quad M = -\frac{EI(\tilde{v}'' + \tilde{w}')}{R} = 0 \tag{6}$$

The linear in-plane elastic buckling load of three-pinned circular arches was obtained by Timoshenko and Gere (1961), and Schmidt (1979) as:

$$q_{cr}R = k_1\frac{EI}{R^2} \tag{7}$$

The values of the factor k_1 are given in Table 1:

Table 1. Values of factor k_1.

Θ (degree)	30	60	90	120	150	180
k_1	108	27.6	12.0	6.75	4.32	3.00

3 NON-LINEAR EQUILIBRIUM

Because the effect of the crown-pin has to be considered, the half arch shown in Figure 1 is used to derive the differential equations of equilibrium for non-linear analysis, by using the principle of virtual work, which states that:

$$\delta\Pi = \int_0^{\Theta} [-NR(\delta\tilde{w}' - \delta\tilde{v} + \tilde{v}'\delta\tilde{v}'\} - M\delta\tilde{v}'' - qR^2]\mathrm{d}\theta = 0 \tag{8}$$

where N and M are the axial compressive force and bending moment; they can be expressed as:

$$N = -AE(\tilde{w}' - \tilde{v} + \frac{1}{2}\tilde{v}'^2) \quad \text{and} \quad M = -EI\frac{\tilde{v}''}{R} \tag{9}$$

Integrating Eq. (8) by parts leads to the differential equations of equilibrium as:

$$N' = 0 \quad \text{and} \quad \frac{\tilde{v}^{iv}}{\mu^2} + \tilde{v}'' = P \quad \text{with} \quad \mu^2 = \frac{NR^2}{EI} \quad \text{and} \quad P = \frac{qR - N}{N} \tag{10}$$

This also leads to the static boundary conditions:

$$\tilde{v}'' = 0 \quad \text{and} \quad \tilde{v}''' + \mu^2\tilde{v}' = 0 \quad \text{at } \theta = 0 \tag{11}$$

and:

$$\tilde{v}'' = 0 \quad \text{at} \quad \theta = \Theta \tag{12}$$

In addition, the essential kinematical boundary conditions are:

$\tilde{w} = 0$ at $\theta = 0$ and $\theta = \Theta$, and $\tilde{v} = 0$ at $\theta = \Theta$

$$\tilde{v} = \frac{P}{\mu^2}\left[\cos\mu\theta - 1 + \mathrm{H}(\theta)\sin\mu\theta\tan\frac{\beta}{2} + \frac{1}{2}\left(\mu^2\theta^2 - \beta^2\right)\right] \quad \text{with} \quad \beta = \mu\Theta \tag{14}$$

The non-linear bending moment M can then be obtained by substituting Eq. (14) as:

$$M = -\frac{EI\tilde{v}''}{R} = -\frac{EIP}{R}\left[1 - \cos\mu\theta - \mathrm{H}(\theta)\sin\mu\theta\tan\frac{\beta}{2}\right] \tag{15}$$

Substituting the solution given by Eq. (14) into the first of Eq. (9), and integrating both sides of the equation over the arch length, leads to an equilibrium equation between the dimensionless load P and the dimensionless axial compressive force parameter β as:

$$A_1P^2 + A_2P + A_3 = 0 \tag{16}$$

where the coefficients A_1, A_2, and A_3 are given by:

$$A_1 = \frac{1}{\beta^2}\left[1 - \frac{5\sin\beta - \beta}{2\beta(1+\cos\beta)}\right] + \frac{1}{6},\quad A_2 = \frac{1}{\beta^2}\left[1 - \frac{2\sin\beta}{\beta(1+\cos\beta)}\right] + \frac{1}{3} \tag{17}$$

$$A_3 = \frac{\beta^2}{\lambda^2},\quad \lambda = \frac{\Theta S}{2r_x} \tag{18}$$

4 COMPARISONS OF LINEAR AND NON-LINEAR ANALYSES

Distributions of non-linear radial displacements along the arch length given by Eq. (14) are compared with their linear counterparts given by Eq. (4) in Figure 2. It can be seen that the radial displacements predicted by linear analysis are smaller than their non-linear counterparts, particularly, for the shallow three-pinned arch with $2\Theta=34.4^{\circ}$. Hence, use of linear analysis cannot correctly predict the serviceability limit state for three-pinned circular arches.

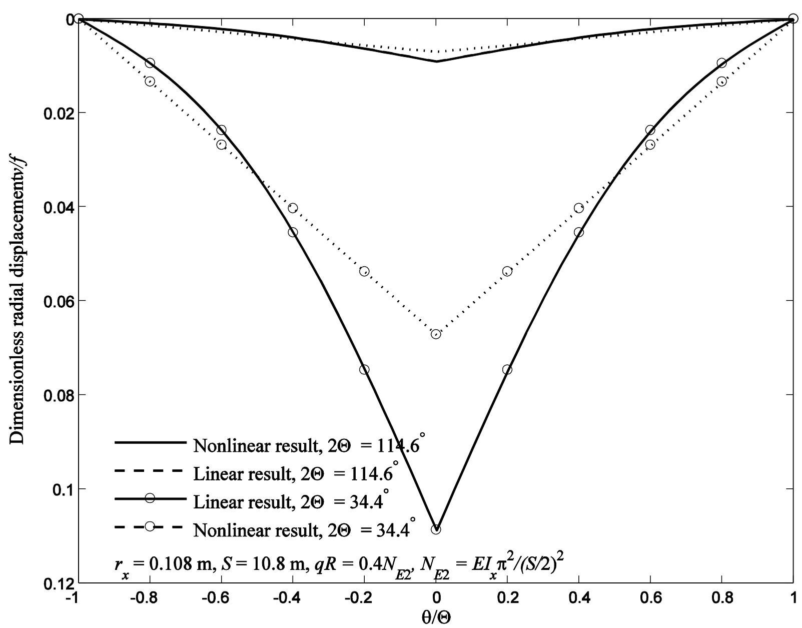

Figure 2. Comparison of distributions of radial displacements.

Linear analysis predicts zero bending moments in three-pinned arches as shown in Eq. (6). However, non-linear analysis predicts bending moments. Typical distributions of bending moments along the arch length given by Eq. (16) are shown in Fig. 3. It can be seen that the uniform radial load produces negative bending moments in three-pinned arches, which are different to the positive bending moments in two-pinned arches produced by the uniform radial load (Pi et al. 2002).

Typical non-linear and linear equilibrium paths are shown in Figure 4. It can be seen from Figure 4 that the linear displacements are much smaller than their non-linear counterparts. It can also be seen from the non-linear results that when the upper limit point is reached, the arch will buckle and snap-through to a stable equilibrium position at the remote equilibrium branch.

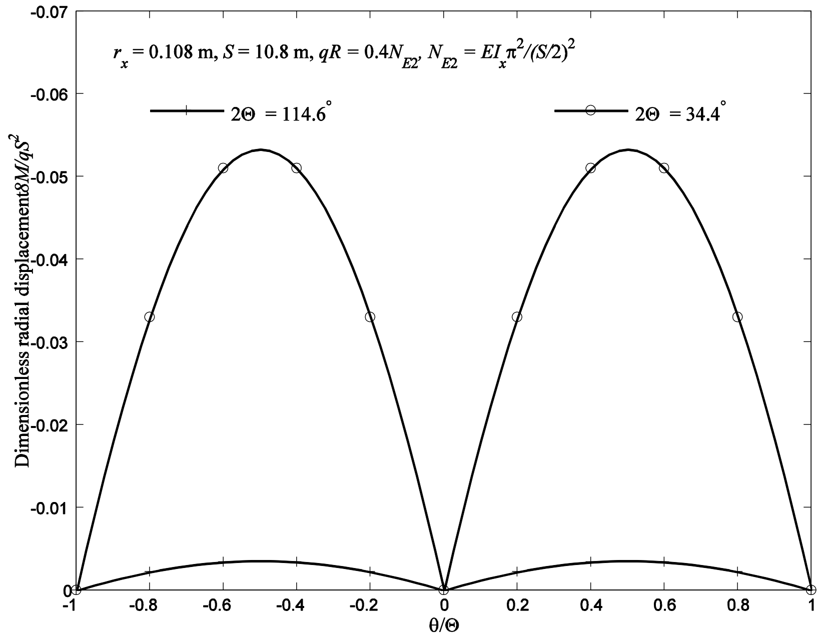

Figure 3. Distributions of non-linear bending moments.

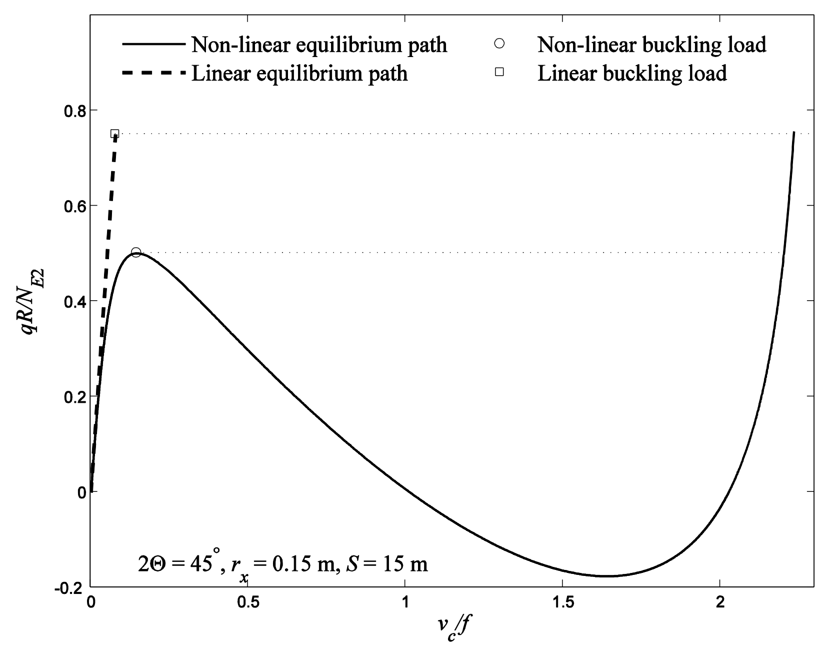

Figure 4. Linear and non-linear equilibrium paths.

5 NON-LINEAR BUCKLING

The non-linear buckling load is the local maximum, so differentiating the non-linear equilibrium equation given by Eq. (16) with respect to β leads to the equilibrium equation between P and β at the limit point as being:

$$B_1P^2 + B_2P + B_3 = 0 \tag{19}$$

where the coefficients B_1, B_2, and B_3 are given by:

$$B_1 = 2A_1 - \frac{\beta\partial A_1}{2\partial\beta}, \quad B_2 = 4A_1, \quad B_3 = A_2 - \frac{\beta_2}{\lambda^2} \tag{20}$$

Solving equations (16) and (19) simultaneously leads to the non-linear limit point buckling load as shown in Figure 4. It can be seen from Figure 4 that the non-linear buckling load is much lower than its linear counterpart.

6 CONCLUSIONS

This paper has shown that the use of linear analysis underestimates the structural response of three-pinned circular steel arches under a radial uniform load, and cannot predict their serviceability limit state correctly. In addition, the linear analytical solutions for the in-plane buckling load of three-pinned circular arches available in the open literature overestimate their buckling loads, and provide unsafe predictions for their in-plane buckling. To predict correct structural in-plane responses and buckling of three-pinned circular steel arches, recourse to non-linear analysis is required.

Acknowledgments

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