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THE STABILITY OF SLOPE USING CELL – BASED SMOOTHED FINITE ELEMENT METHOD AND SECOND ORDER CONE PROGRAMMING

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In this paper, the numerical limit analysis procedure, associating the cell-based smoothed finite element method (CS-FEM) with the (second-order cone) primal-dual interior point algorithm, for cohesive-frictional materials problem is described. The soil is modeled as a cohesionless frictional Mohr-Coulomb material with the associated flow rule. Kinematically admissible velocity fields are established using CS-FEM. The underlying non-smooth optimization problem is formulated as a problem of minimizing a sum of Euclidean norms, ensuring that the resulting optimization problem can be solved by an efficient second order cone programming algorithm. The core purpose of this study is to evaluate collapse loads as well as failure mechanisms of footings on slope which will be obtained directly from solving the optimization problems. In this study, the properties of soil and the width of footing and distance from footing to the edge of the slope are considered. Several numerical examples of slope stability are given to show the performance of the proposed method.

Keywords: Limit analysis, Cohesive-frictional, CS-FEM, SOCP, Optimization, Upper bound.

1 INTRODUCTION

In limit analysis, upper bound and lower bound solution will give the bracket consisting of exactly collapse load. However, the statically admissible stress field is difficult to establish rigorously comparison with the kinematically admissible velocity field. Thus, the upper bound solution normally employed to estimate critical state of structures, especially in geomechanics.

In upper bound limit analysis problems, kinematically admissible velocity fields need be approximated by using a computational method. Once the displacement fields are approximated and the upper bound theorem of plasticity theory applied, limit analysis becomes a problem of optimization involving and can be solved using linear or non-linear programming techniques. This paper proposes a new numerical procedure for kinematic limit analysis problems-the cell-based smoothed element method (CS-FEM) with second-order cone programming-to find out acceptable solutions for some problems in geomechanics such as bearing capacity and slope stability problems (Canh 2009, Canh 2010, Liu 2008, Hung 2009).

2 UPPER BOUND LIMIT ANALYSIS FORMULATION

Consider a rigid-perfectly plastic body of area $\Omega \in R^2$ with boundary Γ, which is subjected to body forces $\boldsymbol{f}$ and to surfaces tractions $\boldsymbol{g}$ on the free portion Γ_t of Γ. The constrained boundary Γ_u

is fixed and $\Gamma_u \cup \Gamma_t = \Gamma$, $\Gamma_u \cap \Gamma_t = \varnothing$. Let $\dot{\boldsymbol{u}} = [\dot{u} \quad \dot{v}]^T$ be plastic velocity or flow fields that belong to a space U of kinematically admissible velocity fields. Where $\dot{u}$ and $\dot{v}$ are the velocity components in the x and y directions respectively. The strain rates can be expressed by relations:

$$\dot{\boldsymbol{\varepsilon}} = \left[\dot{\varepsilon}_{xx} \ \dot{\varepsilon}_{yy} \ \dot{\gamma}_{xy}\right]^T = \nabla \dot{\boldsymbol{u}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}^T \dot{\boldsymbol{u}} \tag{1}$$

The external work rate associated with a virtual plastic flow $\dot{\boldsymbol{u}}$ is expressed in the form as:

$$W_{ext}(\dot{\boldsymbol{u}}) = \int_\Omega \boldsymbol{f}^T \dot{\boldsymbol{u}} d\Omega + \int_{\Gamma_t} \boldsymbol{g}^T \dot{\boldsymbol{u}} d\Gamma \tag{2}$$

The internal plastic dissipation of the two-dimensional domain Ω can be written as: $W_{int}(\dot{\boldsymbol{\varepsilon}}) = \int_\Omega D(\dot{\boldsymbol{\varepsilon}}) d\Omega$, where the plastic dissipation $D(\dot{\boldsymbol{\varepsilon}})$ is defined by $D(\dot{\boldsymbol{\varepsilon}}) = \max_{\psi(\sigma)\le 0} \boldsymbol{\sigma}.\dot{\boldsymbol{\varepsilon}} \equiv \boldsymbol{\sigma}_\varepsilon.\dot{\boldsymbol{\varepsilon}}$, with $\boldsymbol{\sigma}$ represents the admissible stresses contained within the convex yield surface $\psi(\boldsymbol{\sigma})$ represents the stresses on the yield surface associated to any strain rates $\dot{\boldsymbol{\varepsilon}}$ through the plasticity condition.

The kinematic theorem of plasticity states that the structure will collapse if and only if there exists a kinematically admissible displacement field $\dot{\boldsymbol{u}} \in U$, such that:

$$W_{int}(\dot{\boldsymbol{\varepsilon}}) < \lambda^+ W_{ext}(\dot{\boldsymbol{u}}) + W_{ext}^0(\dot{\boldsymbol{u}}) \tag{3}$$

where λ^+ is the collapse load multiplier, $W_{ext}^0(\dot{\boldsymbol{u}})$ is the work of any additional loads $\boldsymbol{f_0}$, $\boldsymbol{g_0}$ not subjected to the multiplier.

If defining $C = \{\dot{\boldsymbol{u}} \in U | W_{ext}(\dot{\boldsymbol{u}}) = 1\}$, the collapse load multiplier λ^+ can be determined by the following mathematical programming:

$$\lambda^+ = \min_{\dot{\boldsymbol{u}} \in C} \int_\Omega D(\dot{\boldsymbol{\varepsilon}}) d\Omega - W_{ext}^0(\dot{\boldsymbol{u}}) \tag{4}$$

3 CELL-BASED SMOOTHED FINITE ELEMENT METHOD (CS-FEM)

In CS-FEM, the problem domain is discretized into elements as in FEM, such as $\Omega = \Omega^1 \cup \Omega^2 \cup ..... \cup \Omega^{nel}$ and $\Omega^i \cap \Omega^j = \varnothing, i \neq j$ the displacement fields are approximated for each element as:

$$u^h(x) = \sum_{I=1}^{n} N_I(x) d_I \tag{5}$$

where n is the number of node per element and $d_I = [u_I v_I]^T$ is the nodal displacement vector.

Elements are then subdivided into several smoothing cells, such as shown in Figure 1, and smoothing operations are performed for each smoothing cell (SC).

A strain smoothing formulation is given by Liu (2006).

$$\begin{aligned} \tilde{\varepsilon}^h(x_C) &= \int_{\Omega^e{}_C} \varepsilon^h(x) \varphi(x, x - x_c) d\Omega \\ &= \int_{\Omega^e{}_C} \nabla u^h(x) \varphi(x, x - x_c) d\Omega \end{aligned} \tag{6}$$

where $\tilde{\varepsilon}^h$ is the smoothed value of strains ε^h for smoothing cell $\Omega^e{}_C$, and φ is a distribution function or a smoothing function that has to satisfy the following properties (Liu 2006): $\varphi > 0$ *and* $\int_{\Omega^e c} \varphi d\Omega = 1$. For simplicity, the smoothing function φ is assumed to be a piecewise constant:

$$\varphi(x, x - x_C) = \begin{cases} 1/A_C, & x \in \Omega^e{}_C \\ 0 \quad , & x \notin \Omega^e{}_C \end{cases} \tag{7}$$

with A_C is the area of the smoothing cell $\Omega^e{}_C$.

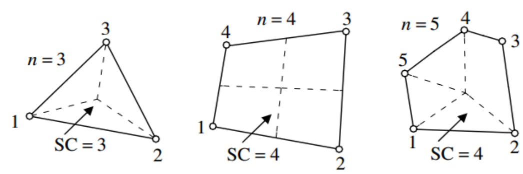

Figure 1. Smoothing cells for various element types: triangular, quadrilateral and polygonal.

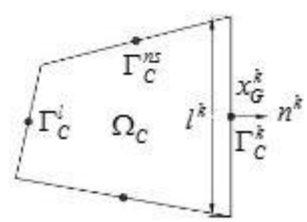

Figure 2. Geometry de finition of a smoothing cell.

Substituting Eq. (6) into Eq. (4), and applying the divergence theorem:

$$\tilde{\varepsilon}^h(x_C) = \frac{1}{A_C}\int_{\Omega^e c} \nabla u^h(x) d\Omega = \oint_{\Gamma_C} n(x) u^h(x) d\Gamma \tag{8}$$

where Γ_c is the boundary of $\Omega^e{}_c$ and n is a matrix with components of the outward surface normal.

The smooth version of the strain rates can be expressed as:

$$\dot{\tilde{\boldsymbol{\varepsilon}}}^{\boldsymbol{h}}(\boldsymbol{x_C}) = \tilde{\boldsymbol{B}}\dot{\boldsymbol{d}} \tag{9}$$

where
$$\dot{\mathbf{d}}^T = [\dot{\mathrm{u}}_1, \dot{\mathrm{v}}_1, \ldots \dot{\mathrm{u}}_n, \dot{\mathrm{v}}_n] \tag{10}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \tilde{N}_{1,x}(x_C) & 0 & \ldots & \tilde{N}_{n,x}(x_C) & 0 \\ 0 & \tilde{N}_{1,y}(x_C) & \ldots & 0 & \tilde{N}_{n,y}(x_C) \\ \tilde{N}_{1,y}(x_C) & \tilde{N}_{1,x}(x_C) & & \tilde{N}_{n,y}(x_C) & \tilde{N}_{n,x}(x_C) \end{bmatrix} \tag{11}$$

with
$$\tilde{N}_{I,\alpha}(x_C) = \frac{1}{A_C}\int_{\Omega^e c} N_I(x) n_\alpha(x) d\Gamma = \frac{1}{A_C}\sum_{k=1}^{ns} N_I(x_G{}^k) n_\alpha{}^k l^k, I = 1, 2, \ldots, n \tag{12}$$

where $\tilde{N}_{I,\alpha}$ is the smoothed version of shape function derivative $N_{I,\alpha}$; n_s is the number of edges of a smoothing cell Ω_c as shown in Figure 2; x_G^k is the Gauss point of Γ_C^k boundary segment which has length l_x and outward surface normal n^k.

4 CS-FEM FORMULATION FOR PLANE STRAIN WITH MOHR-COULOMB YIELD CRITERION

In this study, the Mohr-Coulomb failure criterion is used:

$$\psi(\sigma) = \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\tau_{xy}^{\ 2}} + (\sigma_{xx} + \sigma_{yy}) \sin\varphi - 2c\cos\varphi \tag{13}$$

The plastic strains are assumed to obey the normality rule $\dot{\varepsilon} = \dot{\mu}\dfrac{\partial\psi}{\partial\sigma}$, where the plastic multiplier $\dot{\mu}$ is non-negative.

Hence, the power of dissipation can be formulated as a function of strain rates for each domain i as $D(\dot{\varepsilon}) = cA_i t_i \cos\varphi$.

The upper bound limit analysis for plane strain using the smoothed strains can be formed as:

$$\lambda^+ = \min \sum_{i=1}^{n_{SD} x nel} cA_i t_i \cos\varphi \tag{14}$$

$$s.t \begin{cases} W_{ext}(\dot{\boldsymbol{u}}^h) = 1 \\ \dot{\boldsymbol{u}}^h = 0 \quad on\ \Gamma_u \\ \dot{\tilde{\varepsilon}}_{xx}^h + \dot{\tilde{\varepsilon}}_{yy}^h = t_i \sin\varphi \qquad i = 1, 2, \ldots, nel * n_{SD} \\ \|\boldsymbol{\rho}\|_i \le t_i \qquad\qquad i = 1, 2, \ldots, nel * n_{SD} \end{cases} \tag{15}$$

where n_{SD} is the smoothing cell and *nel* is the number of element in the whole investigated domain. And the fourth constraint in problem (15), resulting optimization problem is cast in the form of a second – order cone programming (SOCP) problem so that a large-scale problem can be solved efficiently (Canh 2009, Canh 2010, Mosek programming).

5 UNDRAINED STABILITY OF FOOTINGS ON SLOPE USING

In this section, a strip footing with a width of B is rested on slope, L is the distance from footing to the edge of the slope, as shown in Figure 3. The ultimate bearing capacity depends on L, the soil unit weight which has the effect on the overall stability of the slope. This is not similar to the surface foundation resting on level ground where soil unit weight has no influence on the ultimate bearing capacity. The ultimate bearing capacity of the considered problem can be stated as:

$$\frac{p}{\gamma B} = f\left(\beta, \frac{L}{B}, \frac{H}{B}, \frac{c_u}{\gamma B}, \frac{q}{\gamma B}\right) \tag{16}$$

with p is the average limit pressure acting on the footing and q is the distributed load.

In this paper, the ratio $H/B = 3$ which ensures that the destruction occurs above the toe of the footing is used. In order to analyse the influence of the footing distance to rest, an angle slope $\beta=90^0$ is considered and $c_u/\gamma B=5$, $q/\gamma B=0$ is established in the limit analysis problem, the footing distance to rest $L/B=0\div4$ is considered so as to compare with result of Shiau (2011) using finite element limit analysis formulations of the upper bound theorem.

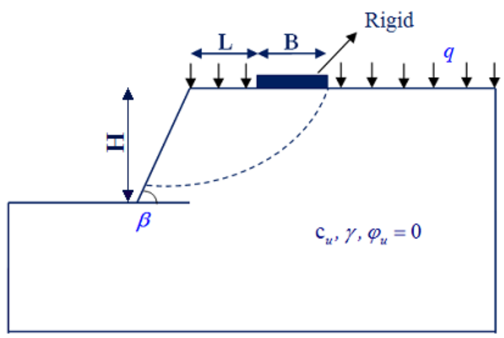

Figure 3. Footing strip resting on slope.

Analysis model using 16000 elements and the result obtained is reported in Table 1.

Table 1. Ultimate bearing capacity (p/γB) for (cu/γB = 5 and q/γB = 0).

	Footing distance to rest L/B				
	0	**1**	**2**	**3**	**4**
CS-FEM	9.5639	15.85	19.21	22.36	24.98
Shiau	9.5	16.12	19.64	22.73	25.35

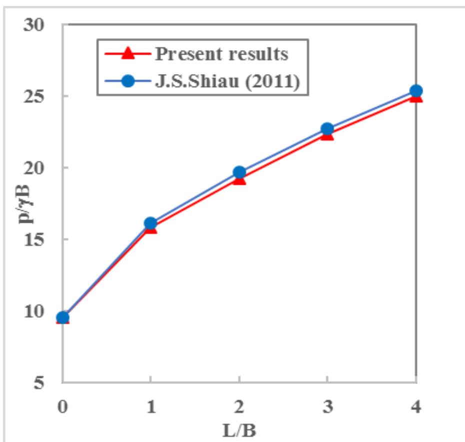

Figure 4. Ultimate bearing capacity (p/γB) with $c_u/\gamma B = 5$ and q/γB = 0.

As shown at Figure 4, the present result is lower (better) than result of Shiau (2011).

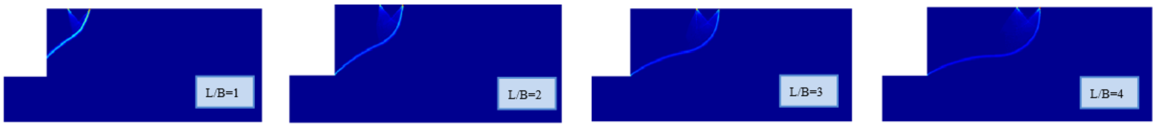

Figure 5. Plastic dissipation distribution with various *L/B* for: $\beta = 90^o$, $cu/\gamma B = 5$, $q/\gamma B = 0$.

5.1 Effect of $c_u/\gamma B$

In the case L/B = 0 with 31600 elements, various ratio of cu/γB from 0.566 to 25 was chosen, a normalized bearing capacity obtained will be shown in Table 2. Although the result of CS-FEM is slightly higher than those of Kusakabe (1981) that using numerical method based on the upper bound theorem of perfect plastic solids, it's better than the result of Shiau (2011) as well as Narita and Yamaguchi's (1990) result based on the limit equilibrium method.

Table 2. Bearing capacity with different cu/γB.

c_u/gB	L/B =0				
	Narita and Yamaguchi (limit equlibrium)	**Kusakabe (upper bound)**	**Shiau (lower bound)**	**Shiau (upper bound)**	**CS-FEM**
25	107	102	97.5	104.33	103.6923
5	21.1	20.2	19.61	20.69	20.5269
1	3.94	3.84	3.73	3.93	3.8578
0.75	-	-	2.59	2.85	2.7687
0.714	-	-	2.36	2.68	2.5403
0.6	-	-	No feasible solution	1.87	1.6116
0.556	-	-	No feasible solution	1.34	1.1137

6 CONCLUSION

A novel procedure for performing upper bound limit analysis using CS-FEM and SOCP has been described. A key advantage of applying the CS-FEM to limit analysis problems is that the size of optimization problem is reduced. Moreover, numerical examples show that when the underlying optimization is cast in the form of a standard SOCP large-scale engineering problems can be solved with a minimal computational cost, and gives acceptable upper bounds for both drained and undrained analysis.

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