

BI-AXIAL STRESS-STRAIN RELATIONS FOR CONCRETE WITH ISOTROPIC AND ANISOTROPIC DAMAGE

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For the load paths with small or no confining pressure, the nucleation, growth, and coalescence of cracks and micro-cracks are the main sources of non-linearity in the observed behavior of concrete. The formations of cracks and micro-cracks destroy material bonds and render the material more compliant. These are typically irreversible internal changes and lead to strong directionality in concrete response. To model such nonlinear material behavior, a constitutive law for concrete utilizing damage mechanics is presented for small and isothermal deformations. The general theory is cast within the framework of the internal variable theory of thermodynamics where the dissipation inequality is used. A damage criterion is subsequently obtained using a damage function and the loading-unloading statement is provided. The decomposition of the compliance tensor into damaged and undamaged states is outlined and the flow rules for the inelastic strains are provided. Specific damage response tensors for isotropic and anisotropic modeling is proposed along with numerical simulations that are plotted for illustrating differences between isotropic and anisotropic formulations.

Keywords: Dissipation, Inequality, Response tensors, Kuhn-Tucker form, Plasticity.

1 INTRODUCTION

Unlike metals where the nonlinearity arises due to the slip along preferential planes and is caused by the presence of local shear stresses, the nonlinear behavior in brittle solids is dependent on the presence or the lack of confining pressures on them. Under low confining pressures, the behavior of brittle solids is influenced and is driven by the formation of micro-cracks and micro-voids. In some brittle solids, such as ceramics, microcracks are already present at grain boundary facets due to cooling processes (Ortiz 1987). Similarly, pre-existing damage zones are documented to exist at the interfaces of aggregates and mortar in concrete. Under far-field stress conditions, micro-cracks that are favorably oriented became active and propagate until arrested by inclusions or phases that possess higher fracture toughness.

At high confining pressures the ductility and strength of brittle materials are enhanced as the formations of micro-cracks and micro-voids are inhibited. Under this loading condition, plastic flow becomes the dominant micro-structural changes and the elastic properties remain practically unchanged (Hueckel and Maier 1977). The photographs produced by Kovari and Tisa (1974) on marble specimen under confining pressures show the slip lines clearly.

To model the inelastic behavior of brittle solids undergoing micro-cracking, damage mechanics theories have been proposed and successfully used (Saboori *et al.* 2014, Saboori *et al.*

2015, Thapa and Yazdani 2014). In this paper, a class of damage mechanics theories where the compliance tensor is regarded as an internal variable is presented in the stress-space. The results of three different damage evolution laws are shown along with an isotropic damage formulation for comparison. The reason for the inclusion of isotropic formulation is that there are situations where damage can remain distributed in a statistical sense and a closer look at isotropic damage formulated is therefore warranted.

2 ANISOTROPIC DAMAGE

To facilitate the formulation that will follow, it is helpful to define a cross-composition operation of tensors, " $\widehat{\otimes}$ ", such that

$$U=V \widehat{\otimes} W = V_{ik} W_{il} e_i \otimes e_i \otimes e_k \otimes e_l$$
(1)

where, the symbol " \otimes " designates tensor product operation, and e_l are orthonormal base vectors. The Cauchy stress tensor, σ , can be expressed by using the eigen-space, as

$$\sigma = \sum_{a=1}^{3} \sigma_{(a)} q_{(a)} \bigotimes q_{(a)}$$
⁽²⁾

where $\mathbf{q}_{\mathbf{a}}$ are the designated eigen-vectors associated with eigen-values σ_a . A positive second-order spectral tensor operator, \mathbf{Q}^+ is defined as

$$Q^{+} = \sum_{a=1}^{3} H(\sigma_{a}) q_{(a)} \otimes q_{(a)}$$
(3)

where H (·) is the Heaviside function. The cross-composition of the tensor \mathbf{Q}^+ by itself yields a fourth-order tensor operator, $\mathbf{P}^+ = \mathbf{Q}^+ \otimes \mathbf{Q}^+$ such that when operated on the stress tensor, it transforms the stress tensor to its positive cone by removing the negative eigen-values; that is

$$\sigma^{+} = \mathbf{P}^{+} : \sigma \tag{4}$$

This elaborate, yet simple, mathematical operation allows for the development of anisotropic damage models as will be shown in the sequel.

In the stress-space formulation, the Gibbs Free Energy function, $G(\sigma, C)$, is used where *C* is the current compliance of the material and changes as damage evolves. Defining the cumulative damage parameter as *k*, the current compliance tensor is expressed as $C(k)=C^0+C^c(k)$ where, C^0 is the undamaged flexibility and C^c reflects the added flexibility due to damage. Clausius-Duhem inequality in the stress space yields that

$$dG(\sigma, \mathbf{C}) \cdot d\sigma : \varepsilon \ge 0 \tag{5}$$

where, $\boldsymbol{\varepsilon}$ is the strain tensor, the differential symbol "d" denotes changes in G and $\boldsymbol{\sigma}$, and ":" denotes a tensor contraction operation. The standard thermodynamics argument produces two results that

$$\boldsymbol{\varepsilon} = \frac{\partial \mathbf{G}}{\partial \sigma} \text{ and } \frac{\partial \mathbf{G}}{\partial \mathbf{C}} :: \mathbf{d} \mathbf{C} \ge 0$$
(6)

where, the first term identifies the Gibbs Free Energy as a thermodynamics potential function for the strain tensor and "::" is the double contraction operation. Since C^0 is not a function of

damage; therefore $dC(k)=dC^{c}(k)$. With this, the second term of Eq. (6) yields the dissipation inequality as (Lubliner 1972)

$$\mathbf{d_s} = \frac{\partial \mathbf{G}}{\partial \mathbf{k}} \mathbf{dk} \ge 0 \tag{7}$$

Damage evolution rules are needed next to capture the effect of cracking on the mechanical properties of brittle solids. To accomplish this we first define a fourth-order tensor, R, such that dC^c=dk R. Different response tensors R would define different damage model. The following forms of R is studied below:

$$\mathbf{R} = \frac{\boldsymbol{\sigma}^{+} \otimes \boldsymbol{\sigma}^{+}}{\boldsymbol{\sigma}^{+} : \boldsymbol{\sigma}^{+}} \tag{8}$$

$$\mathbf{R} = \frac{\boldsymbol{\sigma}^+ \otimes \boldsymbol{\sigma}^+}{\boldsymbol{\sigma}^+ : \boldsymbol{\sigma}^+} + \gamma \ (\mathbf{I} - \mathbf{i} \otimes \mathbf{i}) \tag{9}$$

$$\mathbf{R} = \mathbf{q}_i \otimes \mathbf{q}_j \otimes \mathbf{q}_k \otimes \mathbf{q}_l \tag{10}$$

$$\mathbf{R} = \mathbf{I} = \delta_{ik} \delta_{jl} \, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \tag{11}$$

where, γ is a material parameter, \mathbf{q} , $\sqrt{q_i q_i} = 1$ identifies the direction of maximum tensile stress, the fourth-order identity tensor is denoted by **I**, and **i** represents the second-order identity tensor.

3 DAMAGE CRITERION AND STRAIN COMPONENTS

To progress further, a damage criterion is needed. With an assumption that damage is irreversible, a general form of a damage potential can be obtained as

$$\psi(\boldsymbol{\sigma},\mathbf{k}) = \frac{1}{2}\boldsymbol{\sigma}: \mathbf{R}: \boldsymbol{\sigma} - \frac{1}{2} t^2(\boldsymbol{\sigma},\mathbf{k}) = 0$$
(12)

where, the function, $t(\sigma, k)$, is identified as the damage function to be determined from experimental data from some load path. The damage potential (surface) encompasses an elastic domain in the stress space where when $\psi(\sigma,k) < 0$, the response is elastic. The condition $\psi(\sigma,k)=0$ constitutes the necessary condition for the onset of material inelasticity, and the condition $\psi(\sigma,k) > 0$ is not permitted for rate-independent processes. The loading-unloading statement can then be made using the standard Kuhn-Tucker form as:

$$\psi \leq 0, dk \geq 0 \quad dk \psi = 0$$
 (13)

where, the set of equations given above must be satisfied simultaneously for all admissible processes. The strain components of the deformation are obtained from Eq. (6) such that

$$\boldsymbol{\varepsilon} = \frac{\partial \mathbf{G}}{\partial \boldsymbol{\sigma}} = \mathbf{C}(\mathbf{k}) : \boldsymbol{\sigma}$$
(14)

which represents a set of non-linear equations as the compliance, **C**, evolves with damage. The deformations given in the constitutive relations of Eq. (14) are termed "elastic-damage" processes as no permanent deformation is predicted. Permanent strains are observed in brittle-fracturing processes due to misfit of cracks surfaces, process-zones, etc. but are excluded from consideration in this paper in order to better compare the predictions of isotropic and anisotropic damage

models. To obtain strains, the differential form of Eq. (14) must be used and integrated to arrive at the total strain components. It follows that

$$d\varepsilon = d\varepsilon^{e} + d\varepsilon^{D} = C(k) : d\sigma + dC^{c}: \sigma$$
(15)

in which $d\epsilon^e = C(k) : d\sigma$ is the incremental elastic strain tensor, and $d\epsilon^D = dC^e(k) : \sigma$ denotes incremental damage strains. The form of the damage function used in this paper is obtained from the experimental works of Gopalaratnam and Shah (1985) and proposed by Ortiz (1985) as

$$t(k) = f_t \frac{e \ln \frac{1}{1 + E_0 k}}{1 + E_0 k}$$
(16)

where f_t denotes the uniaxial strength of a brittle solid, E_0 is the initial elastic modulus, and "e" is the natural number.

4 EXAMPLES

The damage function, *t*, is plotted in Figure 1 against the damage parameter *k*. Both axes are normalized for a simpler presentation. The constant k^* is given as $k^* = (e-1)/E_0$ that corresponds to the maximum of the function "t". The form of smooth and differentiable function presented by Eq. (16) is preferred in numerical simulations where no sharp corner conditions are present. Sharp corners have been reported to cause singularities in computational schemes or lead to snap back issues in constitutive algorithms (Cope *et al.* 2005).



Figure 1. Normalized damage function.

Utilizing the differential forms of strain components in Eq. (15), together with the damage criterion of Eq. (12) and using different response tensors, **R**, given in Eq. (8-11), we can obtain the stress-strain relationship for any load paths. Figures 2 and 3 show the predicted stress-strain behavior for axial and lateral strains for equal-biaxial tensile stress path. The uniaxial behavior is also plotted in Figure 2 along with experimental result for comparison. Several points are noted. For one, with the exception of using Eq. (10), the other models predict that under a biaxial tensile loading path, the ultimate strength is lower than the uniaxial strength. This observation is consistent with the experimental results reported elsewhere. To capture the deformational characteristics better, more detailed response tensor along with inelastic damage strain involving kinematic material parameters are required which is beyond the scope of this paper. Nonetheless, it can be seen that the salient features of concrete material inelasticity are captured even with simple damage and flow rules that are considered here.

The apparent Poisson's ratio is also known to change during the process of micro-cracking. This leads to observed dilatation when one studies the volumetric strains versus pressure response. Of the four damage response tensors studied in this paper, only the relation shown in Eq. (9) correctly models this behavior. The other three response tensors do not model any inelastic changes in the lateral direction. For the numerical solutions reported here the following material parameters were used: $E_0 = 4600$ ksi (32000 MPa), v = 0.20; $\gamma = 0.1$, and $f_t = 500$ psi (3.45 MPa).



Figure 2. Normalized uniaxial and equal biaxial stress-strain curves by Eq. (9).



Figure 3. Normalized equal biaxial stress-strain curves.

Other aspects of interest in the formulation to mention are the changes in the elastic moduli in the three orthogonal principal directions for axial and equal biaxial stress paths. The variation of the elastic modulus in the direction of loading for the uniaxial stress path is shown in Figure 4. The figure shows a continuous degradation of the elastic modulus as damage increases. This behavior is supported experimentally when cyclic loading-unloading is performed. Denoting $\mathbf{E} = \mathbf{C}^{-1}$ as the Elasticity tensor, the changes in the components of the compliance tensor \mathbf{C} is worthy of discussion at this point. For equal bi-axial load path, the anisotropic damage rules of Eq. (8) and (9) predict equal changes in components \mathbf{C}^{c}_{1111} and \mathbf{C}^{c}_{2222} and zero damage accumulation in \mathbf{C}^{c}_{3333} . The damage rule given by Eq. (10) assumes added flexibility in the component \mathbf{C}^{c}_{1111} and zero damages in components \mathbf{C}^{c}_{2222} or \mathbf{C}^{c}_{3333} . The isotropic damage formulation of Eq. (11) predicts damage in all three directions affecting \mathbf{C}^{c}_{1111} , \mathbf{C}^{c}_{2222} , and \mathbf{C}^{c}_{3333} components, equally. This could only happen if damage remains isotropic as is sometimes the case under blast loading conditions.



Figure 4. Variation of modulus of elasticity with damage.

5 CONCLUSION

A general damage mechanics theory was laid out in this paper for small, isotropic, and rateindependent processes for brittle solids in general and concrete in particular. The effects of four different damage response tensors were examined on the stress-strain relations and on the changes in the components of the flexibility tensor. It was shown that even though the four damage models would predict similar results in the direction of the loading, only one of the models considered here could predict the changes in the apparent Poisson's ratio. The effect of accumulated damage on the components of compliance, and hence the Elasticity, was also discussed. Of the models studied in this paper, only the isotropic formulation would lead to isotropic state even after damage, while the other three damage rules would simulate anisotropic processes. The permanent deformation was not studied in this paper but its inclusion is straightforward and could be found elsewhere (Yazdani 1993, Yazdani and Karnawat 1996).

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