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FREE VIBRATION OF LINEARLY TAPERED TIMOSHENKO BEAM USING CONSISTENT SHAPE FUNCTIONS

HUONG THANH TRINH[1], KIEN NGUYEN DINH[2], T. HIBINO[3], and BUNTARA S. GAN[3]

[1]*Graduate School of Engineering, Nihon University, Koriyama, Japan*
[2]*Dept of Solid Mechanics, Vietnam Academy of Science and Technology, Hanoi, Vietnam*
[3]*Dept of Architecture, Nihon University, Koriyama, Japan*

Tapered Beam elements are of great importance in a wide range of structural applications because of their optimized distribution of strength and weight compared with the uniform ones. Vibration analysis of beams with variable properties has been receiving great interest from engineers and researchers for a long time. This paper presents the way in which new shape functions are constructed and used for analyzing free vibration of Timoshenko beams with linear variation in height or width and various boundary conditions. Fundamental frequencies from the present work are compared with those obtained by different formulation and approaches. The shape functions in the paper are derived for a solid rectangular beam with linearly taper changing in its sectional dimensions. With the aid of the consistently derived shape functions in the finite element calculation, the solution to vibration problems with the least number of elements can be evaluated with high accuracy. A detailed example is presented and compared with a reference work to illustrate the accuracy and computational efficiency for vibration analysis of linearly tapered Timoshenko beams.

Keywords: FEM, Fundamental frequency, Hamilton principle, Shear locking.

1 INTRODUCTION

Analyzing tapered beams in an efficient and accurate way has attracted much attention from many researchers. These studies cover various aspects of tapered beams in lateral and torsional buckling, shear deformation, and dynamic problems. Since the finite element method is a widely used tool for structural analysis, it is necessary to develop the shape functions for the tapered finite beam element. With the appropriate shape functions for the stiffness matrix, the rotator matrix and loading vectors can be established consistently. Several researchers have proposed the exact stiffness matrices or the shape functions for Euler-Bernoulli beams by solving the differential equations of displacement functions, Tang (1993). Further attempts to find the exact formulations for stiffness matrices and shape functions for the Timoshenko beam have been carried out by some researchers, such as Eisenberger (1985). The latest accomplishment for Timoshenko non-uniform beam formulations makes use of the power series method to derive the dynamic stiffness, Leung (2001), and basic displacement functions, Attarnejad (2010). However, in order to obtain reasonably accurate results, up to 10^{th} order polynomial series functions are required, which causes the formulations to become cumbersome and not concise. In this paper, the beam elements employing consistent formulation of shape functions are derived based on the Hamilton principle requiring only one element for static problems and only two or the least elements for free vibration problems.

2 TIMOSHENKO BEAM THEORY

Strain and kinetic energy for a tapered Timoshenko beam element can be written by

$$U = \frac{1}{2}\int_0^l \left[EA(x)\left(u_{,x}\right)^2 + EI(x)\left(\theta_x\right)^2 + \kappa GA(x)\left(w_{,x} - \theta\right)^2 \right] dx$$
$$T = \frac{1}{2}\int_0^l \left[\rho A(x)\left(u_{,t}^2 + w_{,t}^2\right) + \rho I(x)\theta_{,t}^2 \right] dx \quad (1)$$

Where u, w and θ are the axial displacement, transverse displacement and rotation of the beam; κ is the shear correction factor, E and G are respectively the Young's and shear moduli; ρ is the density; $(.)_{,x}$ and $(.)_{,t}$ represent differentiations with respect to coordinate x and time t, respectively; $A(x)$ and $I(x)$ are respectively the cross section area and moment of inertia linearly vary in the x direction.

3 FORMULATION OF TAPERED BEAM SHAPE FUNCTIONS

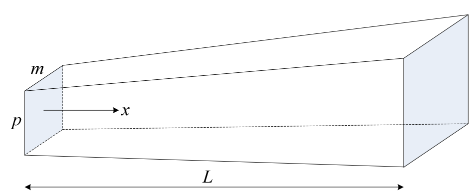


Figure 1. Linearly tapered beam cross-section.

Consider a solid beam with linearly tapered cross-section shown in Figure 1, the variations in height and width of the beam are defined as follows:

$$b(x) = m + n\,x \quad , \quad 0 \le x \le L$$
$$h(x) = p + q\,x \quad , \quad 0 \le x \le L \quad (2)$$

n and q are the width and height taper rates, respectively. From Eq. (1), the Euler-Lagrange equations for the Timoshenko beam are accordingly.

$$\left(EA(x)u_{,x}\right)_{,x} = 0 \quad (3)$$

$$\left(\kappa GA(x)\left(w_{,x} - \theta\right)\right)_{,x} = 0 \quad (4)$$

$$\left(EI(x)\theta_{,x}\right)_{,x} + \kappa GA(x)\left(w_{,x} - \theta\right) = 0 \quad (5)$$

The shape functions for u, w and θ are obtained by solving the differential equations in Eqs. (3-5). The coefficients of the shape functions are defined by substituting the 2nd and 3rd Taylor series for natural logarithmic terms and applying boundary conditions of two end nodes of the beam. For the sake of simplicity, the axial displacement shape functions and its k^{th} order derivatives *(k=0,1,2)* of the linear tapered beam varying in height or width can be given in a series form as α

$$N_{ui} = \frac{1}{\alpha}\sum_{i=1}^{2}\sum_{j=k}^{2}\mathbf{R}_{uij}\left(\frac{x}{L}\right)^{j} \; ; \; \frac{d^k}{dx^k}(N_{ui}) = \frac{1}{\alpha L^k}\sum_{i=1}^{2}\sum_{j=k}^{2}\left\{\frac{j!}{(j-k)!}\mathbf{R}_{uij}\left(\frac{x}{L}\right)^{(j-k)}\right\} \tag{6}$$

where α is a constant defined from taper coefficients (for a uniform section $n=0; q=0; \alpha = 2m^2p^2$) and $\mathbf{R}_{uij}$ is the axial shape function coefficient corresponding to the i^{th} node and the j^{th} degree of freedom.

Similarly, the shape functions for transverse displacement and rotation of the beam and their derivatives can be defined as followed:

$$N_{wi} = \frac{1}{\gamma}\sum_{i=1}^{4}\sum_{j=k}^{3}\mathbf{R}_{wij}\left(\frac{x}{L}\right)^{j} \; ; \; \frac{d^k}{dx^k}(N_{wi}) = \frac{1}{\gamma L^k}\sum_{i=1}^{4}\sum_{j=k}^{3}\left\{\frac{j!}{(j-k)!}\mathbf{R}_{wij}\left(\frac{x}{L}\right)^{(j-k)}\right\} \tag{7}$$

$$N_{\theta i} = \frac{1}{\gamma}\sum_{i=1}^{4}\sum_{j=k}^{2}\mathbf{R}_{\theta ij}\left(\frac{x}{L}\right)^{j} \; ; \; \frac{d^k}{dx^k}(N_{\theta i}) = \frac{1}{\gamma L^k}\sum_{i=1}^{4}\sum_{j=k}^{2}\left\{\frac{j!}{(j-k)!}\mathbf{R}_{\theta ij}\left(\frac{x}{L}\right)^{(j-k)}\right\} \tag{8}$$

γ is a constant defined from taper coefficients (for a uniform section $n=0;\ p=0;\ \gamma = 12(1+\varphi)m^3p^3$); $\mathbf{R}_{wij}$ and $\mathbf{R}_{\theta ij}$ are respectively the transverse and rotation shape function coefficients corresponding to the i^{th} node and the j^{th} degree of freedom. Applying the principle of stationary total energy to Eq. (1) and substituting the derivative shape functions, the following discrete beam element stiffness equations can be obtained in a matrix forms as

$$\mathbf{M}\,\ddot{\mathbf{D}} + \mathbf{K}\,\mathbf{D} = \mathbf{F}_{ex} \tag{9}$$

where, $\mathbf{K}$ and $\mathbf{M}$ is the stiffness matrix and mass matrix of the beam, respectively; F_{ex} is the external nodal load vector. In the free vibration analysis, the right-hand side of the Eq. (9) is set to zero and a harmonic response, $\mathbf{D} = \bar{\mathbf{D}}\sin\omega t$ is assumed so that the Eq. (9) deduced to an eigenvalue problem as

$$\left(\mathbf{K} - \omega^2\,\mathbf{M}\right)\bar{\mathbf{D}} = \mathbf{0} \tag{10}$$

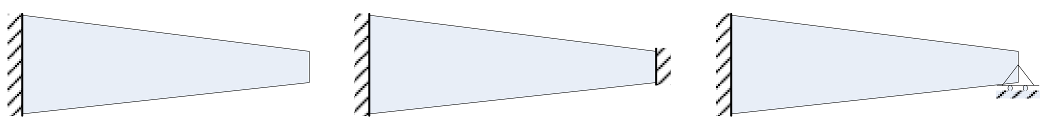
where ω is the circular frequency and $\bar{\mathbf{D}}$ is the vibration amplitude.

$$\mathbf{K} = \int_0^L \begin{bmatrix} [N_{u,x}] \\ [N_{\theta,x}] \\ [N_{w,x}] - [N_\theta] \end{bmatrix}^T \begin{bmatrix} EA(x) & 0 & 0 \\ 0 & EI(x) & 0 \\ 0 & 0 & \kappa GA(x) \end{bmatrix} \begin{Bmatrix} [N_{u,x}] \\ [N_{\theta,x}] \\ [N_{w,x}] - [N_\theta] \end{Bmatrix} dx \tag{11}$$

$$\mathbf{M} = \int_0^l \begin{bmatrix} [N_u] \\ [N_\theta] \\ [N_w] \end{bmatrix}^T \begin{bmatrix} \rho A(x) & 0 & 0 \\ 0 & \rho A(x) & 0 \\ 0 & 0 & \rho I(x) \end{bmatrix} \begin{Bmatrix} [N_u] \\ [N_\theta] \\ [N_w] \end{Bmatrix} dx \tag{12}$$

4 NUMERICAL RESULT

Let consider a tapered beam with $r^2 = I_0/(A_0L^2)$ *=0.01, E/□G=3.12, κ=2/3* made of steel (with Young's modulus *E=210 GPa,* mass densityρ*=7850 kg/m*^{3}) under various boundary conditions (clamped-free, clamped-pinned and clamped-clamped), as shown in Figure 2a-c.



a) Clamped – Free (C - F) b) Clamped – Calmped (C- C) c) Clamped – Pinned (C - P)

Figure 2. Linear – varying height beams with various boundary conditions (BCs).

Table 1. Non-dimensional frequency of tapered beams with various boundary conditions (BCs).

q	BCs	Mode-*i*	μ_1	μ_2	μ_3	μ_4
0	C-F	Attarnejad (2010)	3.2271	14.4689	31.5025	47.9090
		Present	3.2272	14.4778	31.5024	48.1968
		Difference (%)	0.0031	0.0615	-0.0003	0.6007
	C-P	Attarnejad (2010)	11.08250	27.11438	44.84353	59.2030
		Present	11.08680	27.17590	45.11830	59.4452
		Difference (%)	0.0388	0.2269	0.6127	0.4091
	C-C	Attarnejad (2010)	13.83476	28.51793	45.66595	61.8621
		Present	13.83890	28.58680	45.95580	61.4241
		Difference (%)	0.0299	0.2415	0.6347	-0.7080
0.2	C-F	Attarnejad (2010)	3.33065	14.28921	30.71080	47.7502
		Present	3.31530	14.32890	30.91900	48.1303
		Difference (%)	-0.4609	0.2778	0.6779	0.7960
	C-P	Attarnejad (2010)	10.68689	26.10717	43.59072	61.6560
		Present	10.75020	26.31190	44.30880	62.0536
		Difference (%)	0.5924	0.7842	1.6472	0.6449
	C-C	Attarnejad (2010)	13.22227	27.77822	44.69713	61.8066
		Present	13.31860	27.95770	45.12020	61.9055
		Difference (%)	0.7285	0.6461	0.9465	0.1600

The cross-section is rectangular with constant width and linear-varying height. Non-dimensional frequencies of the beam which is defined as $\mu = \omega\sqrt{\rho A_0 L^4 / (EI_0)}$ with two height ratio values (q=0 and q=0.2, q as in Eq. (2)) are given in Table 1 in comparison with that of Attarnejad (2010).

From Table 1, good agreements can be observed among the first four non-dimensional frequencies in the present work by using two elements with the results shown in the reference regardless of boundary conditions and height ratio values.

5 CONCLUSIONS

In the present work, consistent formulation of shape functions for the linearly varying width and height of a Timoshenko beam section were derived based on the Hamilton principle. The beam elements that use these shape functions are "shear locking" free and require only two elements for free vibration problems. The Taylor Series expanded at the right end node was selected to give the smallest errors in the analyses. By using the derived shape functions, in the finite element formulations, the solution of vibration problems with the least number of elements can approximate the results with high accuracy.

Appendix: Coefficient matrices for the shape functions

Define: $A_i = m\ p, A_{jh} = n\ p\ \ell, A_{jb} = m\ q\ \ell, A_{jj} = n\ q\ \ell^2$, where ℓ is the length of the beam element and $\varphi = \dfrac{Ep^2}{\ell^2 \kappa G}$ values at $x = 0$. $\mathbf{A}^T = \begin{bmatrix} A_i & A_{jb} & A_{jh} & A_{jj} \end{bmatrix}$,

$$\mathbf{A}^2 = \begin{bmatrix} A_i^2 & A_{jb}^2 & A_{jh}^2 \end{bmatrix}^T, \ \gamma = \mathbf{A}^T.\begin{bmatrix} 12(1+\varphi) & 13\varphi & 7\varphi \\ -24\varphi & -6\varphi & -8\varphi \\ -12\varphi & -8\varphi & -2\varphi \\ 16\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \ \alpha = \mathbf{A}^T.\begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\mathbf{R}_{u10} = \alpha, \mathbf{R}_{u11} = \mathbf{A}^T.\begin{bmatrix} -2 \\ -4 \\ -4 \\ -6 \end{bmatrix}, \mathbf{R}_{u12} = \mathbf{A}^T.\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{R}_{u20} = 0, \mathbf{R}_{u21} = -\mathbf{R}_{u11}, \mathbf{R}_{u22} = -\mathbf{R}_{u12}$$

$$\mathbf{R}_{w10} = \gamma, \mathbf{R}_{w11} = \mathbf{A}^T.\begin{bmatrix} -12\varphi & 0 & 0 \\ 18\varphi & 0 & 0 \\ 6\varphi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{w12} = \mathbf{A}^T.\begin{bmatrix} -36 & -9\varphi & -3\varphi \\ 6\varphi & 0 & 0 \\ 6\varphi & 0 & 0 \\ -12\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2$$

$$\mathbf{R}_{w13} = \mathbf{A}^T.\begin{bmatrix} 24 & -4\varphi & -4\varphi \\ 0 & 6\varphi & 8\varphi \\ 0 & 8\varphi & 2\varphi \\ -4\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{w20} = 0, \mathbf{R}_{w21} = \mathbf{A}^T.\begin{bmatrix} 6(2+\varphi)\ell & 13\varphi\ell & 7\varphi\ell \\ -12\varphi\ell & -6\varphi\ell & -8\varphi\ell \\ -8\varphi\ell & -8\varphi\ell & -2\varphi\ell \\ 16\varphi\ell & 0 & 0 \end{bmatrix}.\mathbf{A}^2$$

$$\mathbf{R}_{w22} = \mathbf{A}^T.\begin{bmatrix} -6(4+\varphi)\ell & -8\varphi\ell & -4\varphi\ell \\ 6\varphi\ell & 0 & 0 \\ 6\varphi\ell & 0 & 0 \\ -10\varphi\ell & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{w23} = \mathbf{A}^T.\begin{bmatrix} 12\ell & -5\varphi\ell & -3\varphi\ell \\ 6\varphi\ell & 6\varphi\ell & 8\varphi\ell \\ 2\varphi\ell & 8\varphi\ell & 2\varphi\ell \\ -6\varphi\ell & 0 & 0 \end{bmatrix}.\mathbf{A}^2$$

$$\mathbf{R}_{w30} = 0, \mathbf{R}_{w31} = -\mathbf{R}_{w11}, \mathbf{R}_{w32} = -\mathbf{R}_{w12}, \mathbf{R}_{w33} = -\mathbf{R}_{w13}, \mathbf{R}_{w40} = 0$$

$$\mathbf{R}_{w41} = \mathbf{A}^T.\begin{bmatrix} -6\varphi\ell & 0 & 0 \\ 6\varphi\ell & 0 & 0 \\ 2\varphi\ell & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{w42} = \mathbf{A}^T.\begin{bmatrix} 6(-2+\varphi)\ell & -\varphi\ell & \varphi\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2\varphi\ell & 0 & 0 \end{bmatrix}.\mathbf{A}^2$$

$$\mathbf{R}_{w43} = \mathbf{A}^T.\begin{bmatrix} 12\ell & \varphi\ell & -\varphi\ell \\ -6\varphi\ell & 0 & 0 \\ -2\varphi\ell & 0 & 0 \\ 2\varphi\ell & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{\theta 21} = \mathbf{A}^T.\begin{bmatrix} -12(4+\varphi) & -4\varphi & -4\varphi \\ 6\varphi & 0 & 0 \\ 6\varphi & 0 & 0 \\ -4\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2$$

$$\mathbf{R}_{\theta 10} = 0, \mathbf{R}_{\theta 11} = \mathbf{A}^T.\begin{bmatrix} -72/\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{\theta 12} = -\mathbf{R}_{\theta 11}, \mathbf{R}_{\theta 20} = \gamma$$

$$\mathbf{R}_{\theta 22} = \mathbf{A}^T.\begin{bmatrix} 36 & -9\varphi & -3\varphi \\ 18\varphi & 6\varphi & 8\varphi \\ 6\varphi & 8\varphi & 2\varphi \\ -12\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{\theta 30} = 0, \mathbf{R}_{\theta 31} = -\mathbf{R}_{\theta 11}, \mathbf{R}_{\theta 32} = \mathbf{R}_{\theta 11}$$

$$\mathbf{R}_{\theta 40} = 0, \mathbf{R}_{\theta 41} = \mathbf{A}^T.\begin{bmatrix} 12(-2+\varphi) & 4\varphi & 4\varphi \\ -6\varphi & 0 & 0 \\ -6\varphi & 0 & 0 \\ 4\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2, \mathbf{R}_{\theta 42} = \mathbf{A}^T.\begin{bmatrix} 36 & 9\varphi & 3\varphi \\ -18\varphi & -6\varphi & -8\varphi \\ -6\varphi & -8\varphi & -2\varphi \\ 12\varphi & 0 & 0 \end{bmatrix}.\mathbf{A}^2$$

References

Attarnejad, R., Semnani, J., and Shahba, A.,Basic Displacement Functions for Free Vibration Analysis of Non-prismatic Timoshenko Beams, *Finite Elements in Analysis and Design*, Elsevier, 46(10), 916-929, Oct, 2010.

Eisenberger, M., Explicit Stiffness Matrices for Non-prismatic Members, *Computers & Structures*, Elsevier, 20(4), 715-720, Dec, 1985.

Leung, A.Y.T., Zhou, W.E., Lim, C.W., Yuen, R.K.K., and Lee, U., Dynamic Stiffness For Piecewise Non-uniform Timoshenko Column By Power Series - Part I: Conservative Axial Force, *Int. J. Num. Met. Eng*, Elsevier, 51, 505-529, Jun, 2001.

Tang, X. D., Shape Functions Of Tapered Beam-column Elements, *Computers & Structures*, Elsevier, 46(5), 943-953, Mar, 1993.