Implementing Innovative Ideas in Structural Engineering and Project Management
Edited by Saha, S., Zhang, Y., Yazdani, S., and Singh, A.

ISBN: 978-0-9960437-1-7

STRESS FUNCTION OF SIMPLY SUPPORTED CONCRETE BEAM WITH A FRACTURE PROCESS ZONE UNDER UNIFORM LOAD

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Based on the weight integration to obtain the closed solution of cohesive crack problem, a method is proposed to obtain the stress function of a simply supported beam under uniform distributed forces in this paper. The key technique is to determine the weight of several solutions of elastic mechanics problems to satisfy the given crack traction within the cohesive crack surfaces and the boundary conditions. The error degree of the function to satisfy the boundary conditions mainly depends on the number and the location of the selected points. In the formed fracture process zone, there is both the finite magnitude of stress concentration and the smooth closed- crack opening displacement. The FE numerical simulation is also carried out; its results are in good agreement with the present theoretical calculation results.

Keywords: Concrete, Fracture mechanics, Displacement function, Weighted function, Tensile strain softening curve.

1 INTRODUCTION

The fictitious crack model (Hillerborg *et al.* 1976) is generally accepted to explain the fracture characteristics of brittle or quasi-brittle materials. According to the model, there is a fracture process zone (FPZ) at the crack tip, where the cohesive stress σ is interrelated with the crack opening displacement (COD) *w* and σ will drop from f_t as the crack opening increases. The real crack will propagate if the cohesive stress drops to zero corresponding with the critical crack opening. The correlation curve of σ to *w* is named as tensile strain softening curve (TSC).

It is difficult to obtain analytical solution of FPZ. A type of theoretical solution to express the cohesive crack was proposed by Duan and Nakagawa (1988) by the weight integration method, there is both the finite stress concentration and the smoothed crack opening shape within the FPZ at the crack tip. It is used to simulation the fracture process of a four point bending concrete beam (Fujii *et al.* 1991), over consolidated clay (Duan *et al*. 1991) and an unsymmetrical fracture process zone in longitudinal shear deformation (Zhu *et al*. 1997). In this paper, the stress function is obtained to a simply supported concrete beam with FPZ under uniform distributed forces.

2 STRESS FUNCTION BASED ON THE DUAN AND NAKAGAWA MODEL

Considering an elastic plane, complex stress function $F(z,a)$ is expressed as:

$$\nabla^2\nabla^2 F(z,a)=0 \tag{1}$$

$$F(z,a)=\bar{z}F_1(z,a)+F_2(z,a) \tag{2}$$

in which, $z=x+iy$, $\bar{z}=x-iy$, a is the crack half length.

Then the components of stress σ_x, σ_y, τ_{xy} and displacement u, v can be stated in the following form:

$$\sigma_x=2F_1'-\bar{z}F_1''-F_2'' \tag{3}$$

$$\sigma_y=2F_1'+\bar{z}F_1''+F_2'' \tag{4}$$

$$\tau_{xy}=-i(\bar{z}F_1''+F_2'') \tag{5}$$

$$2G(u+iv)=\kappa F_1-(\bar{z}F_1''+F_2'') \tag{6}$$

in which, $\kappa=(3-\nu)/(4-\nu)$ for the plane stress state, and $\kappa=3-4\nu$ for the plane strain state, G is the shear modulus, ν is the Poisson's ratio.

To eliminate the stress singularity at the crack tip, a weighted stress function can be constructed as following:

$$Q(z,a,b)=\int_a^{a+b}\rho(t)F(z,t)\,dt \tag{7}$$

in which, $\rho(t)$ is the assumed weight function defined in the interval (a, $a+b$), b is the cohesion crack length, t is integral variable.

3 STRESS FUNCTION OF A SIMPLY SUPPORTED BEAM WITH CRACK

3.1 Solution 1

Consider a mechanical state; its elastic solution is:

$$\phi(z,a)=\sigma_0(z^2+a^2)^{1/2}/2 \tag{8}$$

$$\varphi(z,a)=-\sigma_0 a^2\log(z+(z^2+a^2)^{1/2})/2 \tag{9}$$

The above solution is with the stress singularities at the crack tips. So it cannot be directly used for analysis of the fracture process problems. The stress function with a process zone can be obtained by the weight integral method as follows, in which, the weight function is taken as $\rho=2(a+b-t)/b^2$ for discussion.

$$F_1 = \mathrm{Re}(-i\overline{Z}\phi_1 - i\varphi_1) \tag{10}$$

$$\phi_1 = \frac{\sigma_0}{b^2}(a+b)[t(z^2+t^2)^{1/2} + z^2\log(t+(z^2+t^2)^{1/2})]_{t=a}^{t=a+b} - \frac{\sigma_0}{3b^2}[(z^2+t^2)^{3/2}]_{t=a}^{t=a+b} \tag{11}$$

$$\varphi_1 = \frac{2\sigma_0}{b^2}(a+b)[-\frac{t^2}{3}\log(z+(z^2+t^2)^{1/2}) + \frac{z^3}{6}\log(t+(z^2+t^2)^{1/2})]_{t=a}^{t=a+b} + \frac{2\sigma_0}{b^2}(a+b)[-\frac{zt}{6}(z^2+t^2)^{1/2} + \frac{t^3}{9}]_{t=a}^{t=a+b} + \frac{\sigma_0}{b^2}[\frac{t^3}{4}\log(z+(z^2+t^2)^{1/2}) + \frac{z}{12}(z^2+t^2)^{3/2} - \frac{z^3}{4}(z^2+t^2)^{1/2} - \frac{t^3}{16}]_{t=a}^{t=a+b} \tag{12}$$

3.2 Solution 2

Uniform load acts on the free edge of simply supported beam. By the elastic theory, the components of stress can be express as:

$$\sigma_x = \frac{6q}{h^3}(l^2 - x^2)y + q\frac{y}{h}(4\frac{y^2}{h^2} - \frac{3}{5}) \tag{13}$$

$$\tau_{xy} = -\frac{6q}{h^3}x(\frac{h^2}{4} - y^2) \tag{14}$$

in which, h and l are the height and span of beam, q is distributed load intensity.

3.3 Proposed Fracture Model

Based on above solution, by superimposing several solutions of elastic mechanics, the stress function of simply supposed beam with crack under uniform load is obtained, as shown in Figure 1. First, make the following assumptions:

1) Considered that the effective height of section-cross includes ligament length and the length of fracture process zone.
2) Specify the propagation direction which is forward along the crack extension.
3) Considered that the crack propagation when the stress reaches the maximum tensile strength at the tip of crack, $\Sigma \sigma_a = f_t$, as shown in Figure 2.
4) Ensure that the force is equal to zero within the scope of the effective height of cross section, $\Sigma T_{1j} = 0$.

5) Satisfy that the moments and the tensile forces are zero at the end of the beam, $\Sigma M_j = 0$, $\Sigma T_{2j} = 0$, and the traction is free at the bottom of the beam, $\Sigma T_{3j} = 0$, $\Sigma Q_j = 0$.

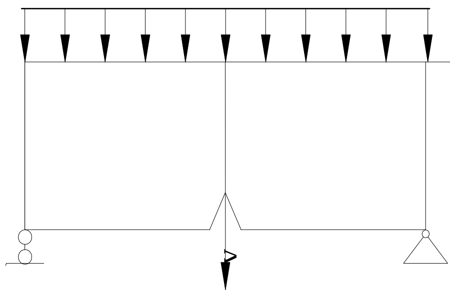

Figure 1. Simply supported beam under uniform load.

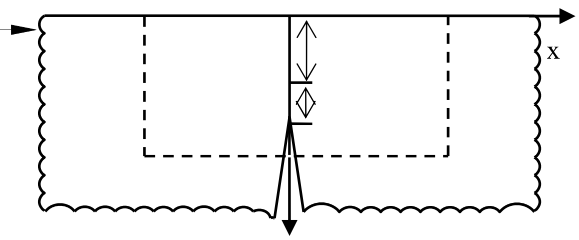

Figure 2. Proposed fracture model.

Base on the above assumptions, define the matrix equation as:

$$\begin{bmatrix} \sigma_{1a} & \sigma_{2a} \cdots\cdots \sigma_{6a} \\ Q_1 & Q_2 \cdots\cdots\cdots Q_6 \\ T_{11} & T_{12} \cdots\cdots T_{16} \\ M_1 & M_2 \cdots\cdots M_6 \\ T_{21} & T_{22} \cdots\cdots T_{26} \\ T_{31} & T_{32} \cdots\cdots T_{36} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} f_t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

then, the coefficients X_i can be yielded out.

4 EXAMPLES

A simply supported concrete beam, h = 8 cm, width b = 7.5 cm, length L = 30 cm, crack width c = 1 cm, concrete tensile strength $f_t = 5.8MPa$, Poisson's ratio $\upsilon = 0.2$, elasticity modulus $E = 28MPa$. The normalized stress distribution along the ligament is demonstrated in Figure 3, the maximum of stress appears at the crack tip, reaching the tensile strength of concrete. Although the total force in the physical crack region is equal to zero, the stress traction is not free everywhere, due to the selected point method is used to satisfy the boundary conditions, including the crack surface in this study. The crack opening displacement and the TSC obtained are as shown in Figure 4 and Figure 5 respectively.

5 NUMERICAL SIMULATION

A FE simulation model is established by the software ANSYS, which contains real cracks and FPZ for the defined beam. It adopts the elastic plane element to divide quadrilateral mesh with the size of 0.1 cm × 0.1 cm. The axial springs are set to the nodes within FPZ, in which the spring stiffness is determined from Figure 5. The simulation results are compared with the theoretical calculation results which are the stress distribution along the typical cross section, as shown in Figure 6.

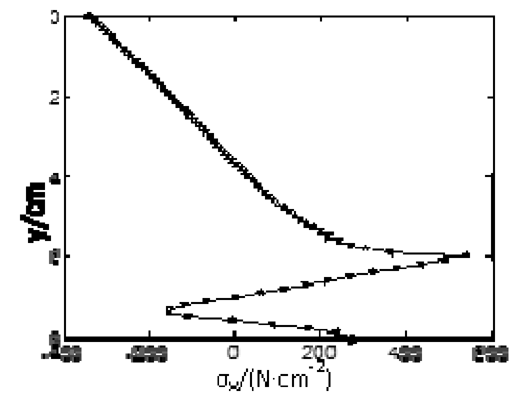

Figure 3. Normal stress along ligament.

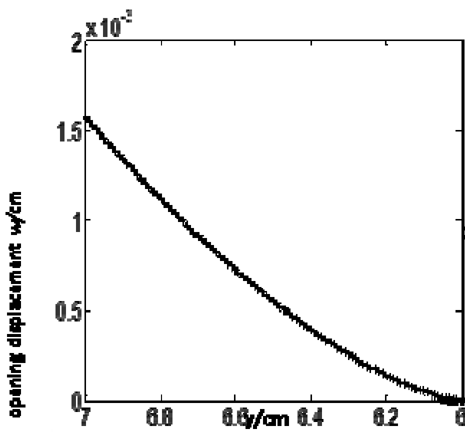

Figure 4. Crack opening displacement.

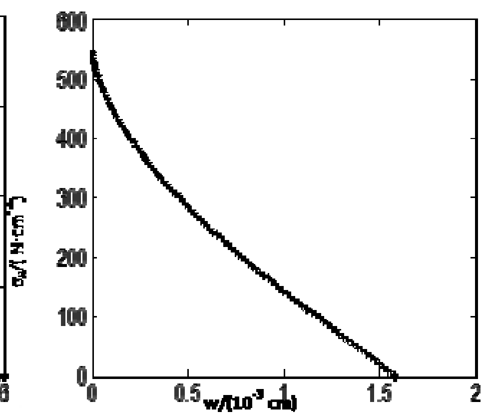

Figure 5. TSC.

6 CONCLUSIONS

The stress function for a simply supported beam with a FPZ is obtained, in which, both the finite magnitude of stress concentration and the smooth closed crack opening displacement appears, the corresponding tensile strain softening curve is depicted. The error degree of the function to satisfy the boundary conditions mainly depends on the number and the location of the selected points. The FE simulation results are in good agreement with the present theoretical calculation results.

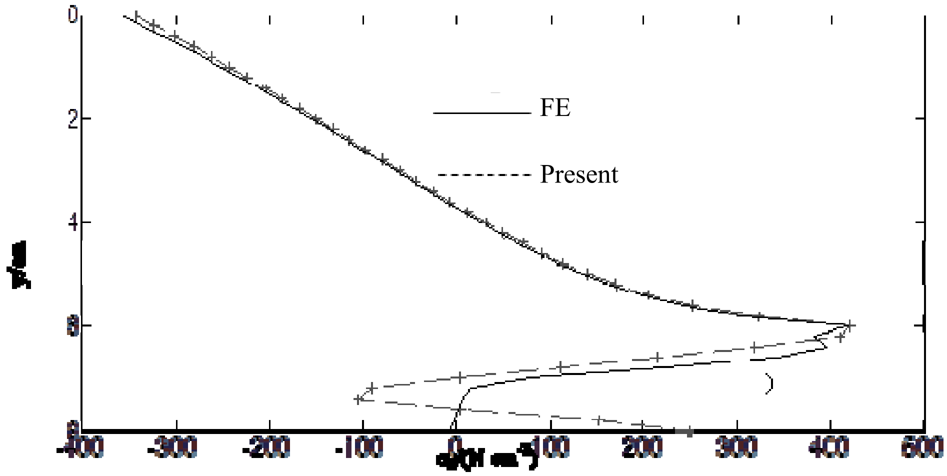

Figure 6. Comparison of theoretical and FE results.

Acknowledgements

The work in this paper was supported by the Nature Science Foundation of Hebei Province, China(Contract No A2015210029) and the key project of Natural Science Foundation of Hebei Education Department, China(Contact No ZH2012040).

References

Duan, S. J., and Nakagawa, K., Stress functions with finite stress concentration at the crack tips for central cracked panel, *Engng Fracture Mech,* 29(5), 517-526, 1988.

Duan, S. J., Nakagawa, K., and Sakaida, T., A mathematical model to approach the fracture process of over-consolidated clay[J], *Engng Fracture Mech*, 38(6), 361-369, 1991.

Fujii, K., Duan, S. J., and Nakagawa, K., A mathematical model for fracture process of four point bending concrete beam, *Engng Fracture Mec*h, 40(1), 37-41, 1991.

Hillerborg, A., Modeer, M., and Petersson, P. E., Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements, *Cement and Concrete Research*, 6(6), 773-782, 1976.

Zhu, M., and Chang, W. V., An unsymmetrical fracture process zone model and its application to the problem of radical crack with an inclusion in longitudinal shear deformation. *Fracture Mechanics of Concrete Structures*, Mihashi, H., and Rokugo, K. (Eds.), 1097-1106, AEDIFICATIO Publishers, Germany, 1998.