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INNOVATIVE IMPLICIT FINITE DIFFERENCE SOLUTION TO TIME RATE OF SETTLEMENT IN CLAY

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The primary consolidation process in clay involves changes in excess pore water pressure (EPWP) with time. The changes in the EPWP are related to permeability, void ratio, and boundary conditions. Closed-form solutions for simple initial EPWP have been published. Such solutions include simplifying assumptions and limited to few impractical initial EPWP distributions. Generally, the initial EPWP encountered in practice is complex, in which case, numerical methods are used. The finite differences and the finite element methods are the primary tools employed in the analysis of settlement behavior of fine-grained soils. Unfortunately, these methods suffer from three main problems: (1) they require evaluation of EPWP vectors at each time increment which results in a large number of calculations; (2) roundoff errors at each time increment; and (3) the solution for a given initial EPWP distribution is achieved at a predetermined integer time increment. An innovative explicit finite difference model is proposed that will permit any initial EPWP distribution, substantially reduces the number of calculations and roundoff errors.

Keywords: Consolidation, Closed-form solution, Excess pore water pressure, Clay soil, Fine-grained soils, Permeability, Roundoff errors.

1 OVERVIEW

Most time rate of settlement estimates are based on a one-dimensional model using an initial applied vertical stress distribution. Consolidation tests are used for the purpose of determining the compression properties of the soil(s) in question. The primary consolidation process involves changes in excess pore water pressure (EPWP) and consequently in void ratio with time. For sand, gravel, and other highly permeable soils, the time required for complete primary consolidation is insignificant when compared to that of clay. The one-dimensional model, which relates EPWP distributions to depth at any time (Terzaghi 1943) is given as:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \tag{1}$$

where u is the excess pore water pressure at time t and depth z and c_V is the coefficient of consolidation. Terzaghi (1943) presented a closed-form solution to this problem that included several simplifying assumptions was applied to a few initial EPWP distributions with depth. For most practical problems, the initial EPWP distribution is non-linear with depth due to partial EPWP dissipation under previous load and/or complex surface load configurations. Consequently, numerical methods are generally used for time rate of settlement calculations.

2 THE FINITE DIFFERENCE METHOD

Besides the finite elements, the finite difference method is perhaps the main numerical technique employed in geotechnical engineering. This method has been used extensively in the solution of time rate of settlement problems (Barden and Berry 1965, Hanson and Nielson 1965, Harr 1966, Hansen and Nielson 1965, Desai and Christian 1977). The solution of Eq. (1) for the EPWP may be viewed as a surface in a three-dimensional space. Therefore, at a given time $t = t_j$ and depth $z = z_i$, the surface is defined in terms of the function $u(z_i,t_j)$. Where m is the number of depth increments and n is the number of time increments for which EPWP is computed. In general, i = 0, 1, ..., m and j = 0, 1, ..., n. The second derivative appearing in Eq. (1) can be approximated using a central derivative finite difference equation (Al-Khafaji and Tooley 1986). Hence, at node z_i and at $t = t_j$, write:

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{at\ t_j, z_i} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta z)^2} \tag{2a}$$

The approximation of the first derivative of u with respect to time at $t = t_j$ and $z = z_i$ is made using a forward difference approximation as:

$$\left.\frac{\partial u}{\partial z}\right|_{at\ t_j, z_i} = \frac{-u_{i,j} + u_{i,j+1}}{\Delta t} \tag{2b}$$

For a given soil layer of thickness Ho, the depth increment Δz = Ho/m and the time increment $\Delta t = t/n$. Direct substitution of Eq. (2a) and Eq. (2b) into Eq. (1) yield a so-called explicit recurrence formula. The Crank-Nicolson method provides a means of approximating the second derivative in which Δt can be made larger without loss of stability. This method estimates the second derivative appearing in Eq. (1) by taking the average central difference approximations to $\partial^2 u/\partial z^2$ at $t = t_j$ and $t = t_{j+1}$. That is:

$$\frac{-u_{i,j} + u_{i,j+1}}{\Delta t} = c_v \frac{(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})}{2(\Delta z)^2} \tag{3}$$

Denoting $\alpha = c_v\, \Delta t/\Delta z^2$, then simplifying gives:

$$u_{i-1,j+1} + (2-2\alpha)u_{i,j+1} + u_{i+1,j+1} = -u_{i-1,j} - (2\alpha - 2)u_{i,j} + u_{i+1,j} \tag{4}$$

Eq. (4) is the implicit finite difference equation, which is stable for any α–value. Additionally, this method will easily handle incompatibility between the initial and boundary condition at time t=0. Therefore, for a soil layer free-draining at the top and bottom, it is possible to express Eq. (4) in matrix form for the m+1 nodes along the depth z as:

$$\begin{bmatrix} 2+2\alpha & -\alpha & & & \\ -\alpha & 2+2\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 2+2\alpha & -\alpha \\ & & & \alpha & 2+2\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 2-2\alpha & -\alpha & & & \\ -\alpha & 2-2\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 2-2\alpha & -\alpha \\ & & & \alpha & 2-2\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{Bmatrix}_{j} \tag{5}$$

Note that elements not shown are equal to zero. In a compact matrix form:

$$[A]\{u\}_{j+1} = [B]\{u\}_j \tag{6}$$

Eq. (6) represents a set of linear algebraic equation that can be solved for the unknowns EPWP vector at time j+1 in terms of values at time j. It is possible to accomplish that using matrix inversion, which gives:

$$\{u\}_{j+1} = [A]^{-1}[B]\{u\}_j = [C]\{u\}_j \tag{7}$$

where $[C] = [A]^{-1}[B]$ and $[A]^{-1}$ is the inverse of matrix [A]. It is important to emphasize that while this method places no restriction on the α-value, the accuracy of the solution still depends on the choice of time and depth increment being used.

3 THE EIGENPROBLEM METHOD

The basis for this method can be best explained by considering a soil layer with free drainage at the top and the bottom boundaries. Suppose that the EPWP initially at t=0 is given by the vector $\{u\}_o$, then the new values of EPWP at times j=0,...n are computed explicitly using Eq. (7).

$$\{u\}_1 = [C]\{u\}_0 \tag{8}$$

$$\{u\}_2 = [C]\{u\}_1 = [C][C]\{u\}_0 = [C]^2\{u\}_0 \tag{9}$$

$$\{u\}_3 = [C]\{u\}_2 = [C][C]^2\{u\}_0 = [C]^3\{u\}_0 \tag{10}$$

and for the n-th increment:

$$\{u\}_n = [C]^{n-j}\{u\}_j \quad for\ j = 0, \tag{11}$$

Eq. (11) has the advantage over Eq. (7) in that the number of calculations and roundoff errors at any time are minimized. Instead, matrix [C] is raised to the desired power directly without the need for computing intermediate EPWP values. Raising a square matrix to any power is an eigenproblem. The eigenvalues and their corresponding vectors for the [C] matrix are determined by solving the following equation:

$$[C]\{\varphi\} = \lambda\{\varphi\} \tag{12}$$

Where λ represents the eigenvalues of the square matrix [C] whose size is (m-1)x(m-1). This is because there are m+1 nodes along depth z and the EPWP at the top and bottom are eliminated due to free drainage. Hence, the solution for the $\{\phi\}$ vector is achieved by forcing the following determinant to zero to yield the m-1 eigenvalues. Thus:

$$|[C] - \lambda[I]| = 0 \tag{13}$$

Expanding the determinant given by Eq. (13) yields a polynomial of order (m-1) and whose roots are the eigenvalues. The evaluation of the eigenvalues is covered in most textbooks on numerical method. For this special case, it can be shown (Al-Khafaji and Tooley 1986) that the eigenvalues of matrix [C] in Eq. (13) can be calculated using:

$$\lambda_T = \frac{2-4\alpha\left[sin^2\left(\frac{r\pi}{2m}\right)\right]}{2+4\alpha\left[sin^2\left(\frac{r\pi}{2m}\right)\right]} \quad \text{for} \quad r = 1,..., m\text{-}1 \tag{14}$$

Where r is the number of interior nodes for which the eigenvalues are needed. Eq. (14) is to be used only when the consolidating layer is free draining at both boundaries. Substituting each of the eigenvalues into Eq. (12) yields the corresponding eigenvectors $\{\phi\}_1$, $\{\phi\}_2$, ..., $\{\phi\}_{m-1}$.

Note that because free drainage is assumed at the top and bottom of the soil layer, there are m-1 depths (nodes) at which the EPWP needs to be computed. Therefore,

$$[C]\{\varphi\}_1 = \lambda_1 \{\varphi\}_1 \tag{15}$$

$$[C]\{\varphi\}_2 = \lambda_2 \{\varphi\}_2 \tag{16}$$

$$\vdots$$

$$[C]\{\varphi\}_{m-1} = \lambda_{m-1} \{\varphi\}_{m-1} \tag{17}$$

These equations can be expressed more conveniently in the following form:

$$[C][\{\varphi\}_1\{\varphi\}_2 \ldots \{\varphi\}_{m-1}] = [\{\varphi\}_1\{\varphi\}_2 \ldots \{\varphi\}_{m-1}]\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_{m-1}\end{bmatrix} \tag{18}$$

in a compact matrix form, we have:

$$[C][\phi] = [\phi][\lambda] \tag{19}$$

where $[\phi]$ is a square eigenvectors matrix and $[\lambda]$ is a diagonal eigenvalues matrix. Therefore, multiplying Eq. (18) by the inverse of the eigenvector matrix $[\phi]^{-1}$ gives:

$$[C] = [\phi][\lambda][\phi]^{-1} \tag{20}$$

The square of the matrix [A] is now given as:

$$[C]^2 = [\phi][\lambda][\phi]^{-1}[\phi][\lambda][\phi]^{-1} \tag{21}$$

$$[C]^2 = [\phi][\lambda]^2[\phi]^{-1} \tag{22}$$

Similarly raising [C] to the n^{th} power yields:

$$[C]^n = [\phi][\lambda]^n[\phi]^{-1} \tag{23}$$

The advantage of Eq. (23) involves the simple task of raising its diagonal elements to any power including a fraction. Substituting Eq. (23) into Eq. (11) gives the implicit solution to the one-dimensional time rate of settlement problem.

$$[u]_n = [\phi][\lambda]^{n-j}[\phi]^{-1}[u]_j \qquad for \quad j = 0, \ldots n \tag{24}$$

It is often convenient to express time in terms of the dimensionless parameter known as the time factor T. This is expressed in terms of the length of drainage path H_{dp}, the time in question t, and the coefficient of consolidation c_v as follows:

$$T = \frac{C_v t}{H_{dp}^2} \tag{25}$$

The length of drainage path can be expressed in terms of the thickness of the consolidating layer, Ho. For a soil drained at both ends:

$$H_{dp} = \frac{H_o}{2} = \frac{m}{2}\Delta z \tag{26}$$

The coefficient of consolidation can be expressed next in terms of α and the depth and time increments as follows:

$$c_v = \frac{(\Delta z)^2}{\Delta t}\alpha \tag{27}$$

Substituting Eq. (26) and Eq. (27) into Eq. (25) and noting that t = nΔt gives the time factor in terms of α, depth and time increments. Thus,

$$T = \frac{\frac{(\Delta z)^2}{\Delta t}\alpha(n\Delta t)}{\left(\frac{m}{2}\Delta z\right)^2} = \frac{4\alpha n}{m^2} \tag{28}$$

The average degree of consolidation for the entire soil layer at any time U_j is determined in terms of the initial area enclosed by the EPWP versus depth distribution A_o and the area enclosed by the EPWP versus depth at any time A_j. Thus,

$$U_j = 1 - \frac{A_j}{A_o} \quad \text{for} \quad j = 0,\ldots n \tag{29}$$

The areas A_o and A_j may be computed using the numerical methods of integration, such as the trapezoidal rule or Simpson's 1/3 Rule.

4 NUMERICAL EXAMPLE

Determine the average degree of consolidation, time factor, and the EPWP after 2 years for a clay layer with free-draining boundaries assuming H_o = 18 m, c_v = 6.48 m^2/yr, an initial EPWP distribution of 100 kN/m^2 and α = 1/6.

5 SOLUTION

Since the compressible soil layer has free draining boundaries, the length of drainage is taken as H_{dp}= H_o/2 = 9 m. Substituting α = 1/6 and Δz = 18/10 = 1.8 m into Eq. (27), then solving for the time increment gives:

$$\Delta t = \frac{(1/6)(1.8)^2}{6.4} = 0.0844\ year \tag{30}$$

Thus, n = 2/0.0844 = 23.7. Note that the time increment is not an integer but this method works regardless of the calculated time increment! The $\{u\}_o$ vector is now used in Eq. (24) to calculate the EPWP after 2 years. That is substituting j = 0 and n = 30.62 gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_9 \end{Bmatrix}_{23.7} = [\varphi]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}^{23.7} [\varphi]^{-1}\begin{Bmatrix} 100 \\ 100 \\ \vdots \\ 100 \end{Bmatrix}_0 \tag{31}$$

Thus, n = 2/0.05 = 40. The $\{u\}_0$ vector is now used in Eq. (24) to calculate the EPWP after 2 years. That is substituting j = 0 and n = 40 gives:

$$\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_9 \end{Bmatrix}_{40} = [\{\varphi_1\}\{\varphi_2\} \ldots \{\varphi_9\}] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}^{40} \begin{bmatrix} \{\varphi_1\}^t \\ \{\varphi_2\}^t \\ \vdots \\ \{\varphi_9\}^t \end{bmatrix} \begin{Bmatrix} 100 \\ 100 \\ \vdots \\ 100 \end{Bmatrix}_0 \quad (32)$$

Note that $[\varphi], [\varphi]^{-1}$and the eigenvalues matrices are not listed here due to space limitations. The calculated EPWP vector is given as at time t = 2 years and 30.62 time increments$\{u\}^{\text{transpose}}$ = {0, 27.73, 51.85, 69.85, 80.69, 84.28, 80.69, 69.85, 51.85, 27.73, 0}. The time factor is calculated next by substituting m = 10, α=1/6, and n=23.7 into Eq. (28) which gives T= 0.158. The corresponding average degree of consolidation after 2 years is computed next using Eq. (29) and Simpson's 1/3 rule of integration. That is

$$U_{23.7} = 1 - \frac{\left(\frac{1.8}{3}\right)[0+4(27.73)+2(51.85)+\cdots.+0]}{100(18)} = 1 - \frac{988.76}{1800} = 45.07 \quad (33)$$

The calculated time factor and the average degree of consolidation compare to the analytical values of U = 45% and T = 0.159 reported by Perloff and Baron (1975).

6 CONCLUSION

A numerical method for solving the one dimensional consolidation time rate of settlement problem with an arbitrary initial EPWP is presented. Unlike traditional numerical techniques, it is versatile in that once a model is developed, then arbitrary initial EPWP distribution can be analyzed without the need for reworking the problem. Note that using the finite difference, one would be required to solve a given problem for a particular initial EPWP and the solution is not necessarily applicable to other initial EPWP distributions. This is not the case with this approach. Furthermore, the solution procedure presented herein permits the determination of EPWP at any time without the need for computing intermediate EPWP values as required with the traditional finite difference methods. In fact, it can even be used to calculate the EPWP at a fraction of a time increment. This eliminates the need for interpolating when intermediate values between increments are needed. The proposed method reduces substantially the roundoff errors associated with other numerical techniques because the solution at any time is given in terms of an initial EPWP vector.

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