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INNOVATIVE EXPLICIT FINITE DIFFERENCE SOLUTION TO TIME RATE OF SETTLEMENT IN CLAY

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The time rate of settlement process in clay involves changes in excess pore water pressure (EPWP) with time. Exact solutions have been published for constant initial EPWP and other simple distributions. The finite differences methods are generally used in solving complex initial EPWP distributions. Such methods suffer from roundoff errors at each time increment and truncation errors proportional to the step size used. The explicit finite difference method produces stable solutions when proper time and depth increments are used. An innovative explicit finite difference model involving eigenvalues and eigenvectors is proposed that will permit arbitrary initial EPWP distributions and reduce roundoff errors. This method is numerically stable and convergent. Unlike traditional methods, the proposed solution will also eliminate the need to calculate the EPWP vector traditionally required at each time increment. Instead, the EPWP can be computed directly for any number of time increments.

Keywords: Exact solution, Excess water pressure, Clay soil, Roundoff errors, Truncation errors, Implicit finite difference, Eigenvalues, Step size.

1 BACKGROUND

Although analytical methods have and will continue to provide useful solution, they cannot yield realistic answers for problems involving nonhomogeneous and/or anisotropic materials with arbitrary boundary and/or initial conditions. While a complete treatment of this broad subject is beyond the scope of this paper, the one-dimensional time rate of settlement problem with arbitrary initial excess porewater pressure (EPWP) is discussed. The mathematical model relating the u is the excess pore water pressure u to time t, depth z and the soil coefficient of consolidation c_V is was presented by Terzaghi (1943) as follows:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \tag{1}$$

Terzaghi and and Frolich (1936) published exact solutions to this model using simplifying assumptions for constant initial EPWP and other simple distributions with depth. In engineering practice, the initial EPWP distribution is normally non-linear with depth due to complex surface load configurations and EPWP dissipation. Thus, the finite differences methods are typically used for time rate of settlement calculations.

2 THE FINITE DIFFERENCE METHOD

The basic concept involves discretization of arbitrary continuous functions and replacing them with their equivalent difference expressions. The solution of Eq. (1) at a given time $t = t_j$ and depth z_i with an arbitrary function $u(z_i, t_j)$ at node z_i can be approximated using difference expressions available in literature (Al-Khafaji and Tooley 1986) as follows:

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{at\ t_j, z_i} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta z)^2} \tag{2a}$$

The approximation of the first derivative of u with respect to time at $t = t_j$ and $z = z_i$ is made using a forward difference approximation as:

$$\left.\frac{\partial u}{\partial z}\right|_{at\ t_j, z_i} = \frac{-u_{i,j} + u_{i,j+1}}{\Delta t} \tag{2b}$$

Substituting Eq. (2a) and Eq. (2b) into Eq. (1) yields:

$$\frac{-u_{i,j} + u_{i,j+1}}{\Delta t} = c_v \frac{(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})}{(\Delta z)^2} \tag{3}$$

For a given soil layer of thickness Ho, the depth increment $\Delta z = Ho/m$ and the time increment $\Delta t = t/n$. Where m is the number of depth increments and n is the number of time increments for which EPWP is computed. In general, i = 0, 1, ..., m and j = 0, 1, ..., n. Denoting $\alpha = c_V \Delta t/\Delta z^2$, then simplifying and rearranging Eq. (3) yields:

$$u_{i,j+1} = \alpha u_{i-1,j} - (1 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \tag{4}$$

Eq. (4) is an explicit finite difference recurrence formula, which permits direct step-by-step evaluation of the EPWP. The implication is that knowing the initial and boundary EPWP values at t = 0, it is possible to calculate the EPWP for a given time increment $t=\Delta t$. Using Eq. (3), a set of linear algebraic equations can be developed which can be expressed in a matrix form for the m+1 nodes along the depth axis z as follows:

$$\begin{Bmatrix} u_{top} \\ u_1 \\ \vdots \\ u_{m-1} \\ u_{bot} \end{Bmatrix}_{t=t_{j+1}} = \begin{bmatrix} \alpha & 1-2\alpha & \alpha & & & \\ & \alpha & 1-2\alpha & \alpha & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \alpha & 1-2\alpha & \alpha & \\ & & & \alpha & 1-2\alpha & \alpha \end{bmatrix} \begin{Bmatrix} u_{top} \\ u_1 \\ \vdots \\ u_{m-1} \\ u_{bot} \end{Bmatrix}_{t=t_j} \tag{5a}$$

Note that the EPWP drops to zero for doubly drained layer after the first time increment. That is, at $t = t_1$ we have $u_{top} = u_{bot} = 0$ and Eq. (5a) is simplified to the following:

$$\begin{Bmatrix} u_1 \\ \vdots \\ u_{m-1} \end{Bmatrix}_{t=t_{j+1}} = \begin{bmatrix} 1-2\alpha & \alpha & & \\ \alpha & 1-2\alpha & \alpha & \\ \ddots & \ddots & \ddots & \ddots \\ & \alpha & 1-2\alpha & \alpha \\ & & \alpha & 1-2\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_{m-1} \end{Bmatrix}_{t=t_j} \tag{5b}$$

Eq. (5b) can be written more conveniently in a compact matrix form as:

$$\{u\}_{j+1} = [A]\{u\}_j \tag{6}$$

Eq. (6) is the explicit finite difference solution to the time rate of settlement problem. The range of α should be between 0.0 and 0.5 for the solution to be stable, in fact $\alpha = 1/6$ gives the most accurate results (Scott 1963). It is evident that the limitation imposed on α makes it necessary that the EPWP be evaluated at extremely large number of time increments. This is precisely why the finite difference method is time consuming and subject to roundoff errors.

3 THE EIGENPROBLEM METHOD

Suppose that the EPWP initially at t = 0 is given by the vector $\{u\}_O$, then the new values of EPWP at times j = 0,..., n are computed explicitly using Eq. (6):

$$\{u\}_1 = [A]\{u\}_0 \tag{7}$$

$$\{u\}_2 = [A]\{u\}_1 = [A][A]\{u\}_0 = [A]^2\{u\}_0 \tag{8}$$

$$\{u\}_3 = [A]\{u\}_2 = [A][A]^2\{u\}_0 = [A]^3\{u\}_0 \tag{9}$$

and for the n^{th} increment:

$$\{u\}_n = [A]^{n-j}\{u\}_j \text{ for } j = 0, \dots n \tag{10}$$

Eq. (10) minimizes the roundoff errors, truncation errors and error propagation at each time step. This is made possible by raising the matrix [A] to the desired power directly without the computing the EPWP values at each time step as is normally required when using the finite difference methods. The eigenvalues and their corresponding vectors for the [A] matrix can be calculated by solving the following equation:

$$[A]\{\varphi\} = \lambda\{\varphi\} \tag{11}$$

Where λ represents the eigenvalues of the square matrix [A] whose size is (m-1) x (m-1). Note that there are m nodes with depth and the EPWP at the top and bottom boundaries of the soil layer are eliminated assuming free drainage. Thus, the set of eigenvalues and eigenvectors of the matrix [A] are calculated by forcing the determinant below to equal zero.

$$|[A] - \lambda[I]| = 0 \tag{12}$$

Expanding the determinant in Eq. (12) yields a polynomial of order (m-1) and whose roots are the eigenvalues. The evaluation of the eigenvalues is covered in most textbooks on numerical method. Fortunately, for this special case, the eigenvalues are given directly by the following equation:

$$\lambda_T = 1 - 4\alpha\left[\sin^2\left(\frac{r\pi}{2m}\right)\right] \quad \text{for} \quad r = 1,\ldots m\text{-}1 \tag{13}$$

Where r is the interior node along the soil depth for which the eigenvalue is required. Eq. (13) is valid only when the soil layer is free draining at the top and bottom boundaries. Substituting each of the calculated eigenvalues into Eq. (12) yields the corresponding eigenvectors $\{\varphi\}_1$, $\{\varphi\}_2$,...$\{\varphi\}_{m-1}$. The eigenvalues and eigenvectors for matric [A] can now be expressed as follow:

$$\begin{gathered}[A]\{\varphi\}_1 = \lambda_1\{\varphi\}_1 \\ [A]\{\varphi\}_2 = \lambda_2\{\varphi\}_2 \\ \vdots \\ [A]\{\varphi\}_{m-1} = \lambda_{m-1}\{\varphi\}_{m-1}\end{gathered} \tag{14}$$

These equations can be expressed in the following compact matrix form:

$$[A][\{\varphi\}_1\{\varphi\}_2 \ldots \{\varphi\}_{m-1}] = [\{\varphi\}_1\{\varphi\}_2 \ldots \{\varphi\}_{m-1}]\begin{bmatrix}\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{m-1}\end{bmatrix} \tag{15}$$

or more simply:

$$[A][\phi] = [\phi][\lambda] \tag{16}$$

where $[\phi]$ is a square eigenvectors matrix and $[\lambda]$ is a diagonal eigenvalues matrix. Therefore, multiplying Eq. (16) by the inverse of the eigenvector matrix $[\phi]^{-1}$ gives:

$$[A] = [\phi][\lambda][\phi]^{-1} \tag{17}$$

The square of the matrix [A] is now given as:

$$[A]^2 = [\phi][\lambda][\phi]^{-1}[\phi][\lambda][\phi]^{-1} = [\phi][\lambda]^2[\phi]^{-1} \tag{18}$$

Similarly raising [A] to the n^{th} power yields:

$$[A]^n = [\phi][\lambda]^n[\phi]^{-1} \tag{19}$$

Substituting Eq. (16 19) into Eq. (10) gives the implicit solution to the one-dimensional time rate of settlement problem:

$$[u]_n = [\phi][\lambda]^{n-j}[\phi]^{-1}[u]_j \qquad \text{for} \quad j = 0, \ldots n \tag{20}$$

Eq. (20) involves multiplication of three matrices, regardless of the number of time increment j. This is true since raising a diagonal matrix to the power n^{th} power is accomplished by raising its diagonal values to the power n. This is precisely the advantage of using Eq. (20) over the conventional finite difference procedure.

The time parameter t is normally expressed in terms of a dimensionless time factor referred to as the time factor T. Since length of drainage path H_{dp}, the time in question t, and the coefficient of consolidation c_v, we write:

$$T = \frac{C_v t}{H_{dp}^2} \tag{21}$$

The length of drainage path can be expressed in terms of the thickness of the consolidating layer, H_o. For a soil drained at both ends:

$$H_{dp} = \frac{H_o}{2} = \frac{m}{2}\Delta z \tag{22}$$

The coefficient of consolidation can be expressed next in terms of α and the depth and time increments as follows:

$$c_v = \frac{(\Delta z)^2}{\Delta t}\alpha \tag{23}$$

Substituting Eq. (22) and Eq. (23) into Eq. (21) and noting that t = nΔt gives the time factor in terms of α, depth and time increments. Thus:

$$T = \frac{\frac{(\Delta z)^2}{\Delta t}\alpha(n\Delta t)}{\left(\frac{m}{2}\Delta z\right)^2} = \frac{4\alpha n}{m^2} \tag{24}$$

The average degree of consolidation for the entire soil layer at any time U_j is determined in terms of the initial area enclosed by the EPWP versus depth distribution A_o and the area enclosed by the EPWP versus depth at any time A_j. Thus,

$$U_j = 1 - \frac{A_j}{A_o} \quad \text{for} \quad j = 0,\ldots n \tag{25}$$

The areas A_o and A_j are computed using the Simpson 1/3 rule of integration at any time t_j.

4 NUMERICAL EXAMPLE

Determine the average degree of consolidation, time factor, and the EPWP after 5 years for a doubly drained clay layer with a thickness of 18 m, c_V = 15.0 m^2/yr, and an initial EPWP distribution of 100 kN/m^2. Assume α = 1/6 and six depth increments.

Solution: Since H_{dp}= H_o/2 = 9.0 m and substituting α=1/6 and Δz = 18/6 = 3.0 m into Eq. (25) gives:

$$\Delta t = \left(\frac{1}{6}\right)\frac{(3)^2}{15} = 0.10 \text{ year} \tag{26}$$

The number of time increments is computed as n = 5/0.10 = 50. The EPWP at the boundaries drops to zero at t > 0. Furthermore, at t = 0 the EPWP u_{top} = u_{bot} = 100 kN/m^2. This discrepancy between the boundary conditions is resolved by taking the average value of EPWP of 50 kN/m^2 as the initial EPWP at the boundaries. Thus:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_1 = \begin{bmatrix} 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 2/3 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 100 \\ 100 \\ 50 \end{bmatrix}_0 = \begin{Bmatrix} 91.67 \\ 100.00 \\ 100.00 \\ 100.00 \\ 91.67 \end{Bmatrix}_1 \tag{27}$$

The eigenvalues and eigenvectors for this problem is given below and the $\{u\}_1$ vector can be calculated after 5 years directly without intermediate steps. That is substituting j = 1 and n = 50 into Eq. (27) gives:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_{50} = \frac{1}{12}\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ \sqrt{3} & -1 & 0 & 1 & -\sqrt{3} \\ 2 & 0 & -1 & 0 & 2 \\ \sqrt{3} & -1 & 0 & -1 & -\sqrt{3} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9533 & 0 & 0 & 0 & 0 \\ 0 & 5/6 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0.3780 \end{bmatrix}^{49} \begin{bmatrix} 1 & \sqrt{3} & 2 & \sqrt{3} & 1 \\ -3 & -1-3 & 0 & 3 & 3 \\ 4 & 0 & -4 & 0 & 4 \\ -3 & 3 & 0 & -3 & 3 \\ 1 & -\sqrt{3} & 2 & -\sqrt{3} & 1 \end{bmatrix} \begin{Bmatrix} 91.67 \\ 100.00 \\ 100.00 \\ 100.00 \\ 91.67 \end{Bmatrix}_1 \tag{28}$$

The calculated EPWP vector is given as $\{u\}_{50}$ = {0, 6.469, 11.205, 12.938, 11.205, 6.469, 0}. The time factor is calculated next by substituting m = 6, α = 1/6, and n = 50 into Eq. (24) which gives T = 0.926. The corresponding average degree of consolidation is computed next using Eq. (19) and Simpson's 1/3 rule of integration. That is:

$$U_{50} = 1 - \frac{\left(\frac{3.0}{3}\right)[0+4(6.469)+2(11.205)+\cdots.+0]}{100(18)} = 91.76\% \quad (29)$$

The time factor and the average degree of consolidation compare rather closely with the analytical values of T = 0.90 and U = 91.20% reported by Perloff and Baron (1976).

5 CONCLUSION

An innovative technique for solving the time rate of settlement in fine-grained soils is presented. The new technique is efficient and versatile in that it applies to any initial EPWP. Unlike traditional finite differences methods, once the solution is achieved for a set of interior nodes, then arbitrary initial EPWP distributions are handled without the need to use finite differences. The proposed method produces stable solutions when proper time and depth increments are used. The formulation involves eigenvalues and eigenvectors that reduce roundoff and truncation errors. Additionally, the proposed model allows the determination of EPWP at any time without the need for computing EPWP values at each time step. It is also possible to calculate the EPWP at a fraction of a time increment which eliminates the need for interpolation. This is the case when dealing with the finite difference procedure. This innovative method reduces substantially the roundoff error and computational time associated with other numerical techniques. The solution at any time is given in terms of an initial EPWP vector and is achieved by multiplying three matrices. This method is numerically stable and convergent.

References

Al-Khafaji, A. W. N.,and Tooley, J. R., *Numerical Methods in Engineering Practice*, Holt, Rinehart and Winston, Inc., New York, 1986.

Perloff, W. H. and Baron, *Soil Mechanics*, The Ronald Press Company, New York, 1976.

Scott, R. F., *Principles of Soil Mechanics*, Addison Wesley Publishing Company, Inc., Reading, Massachusetts, 1963.

Terzaghi, K., *Theoretical Soil Mechanics*, Wiley, New York, 1943.

Terzaghi, K. and Frolich, O. K., *Theorie der Setzung von Tonschichten: Eine Einfuhrung in die Analytische Tonmechanik*, Deuticke, Leipzig, 1936.