Resilient Structures and Sustainable Construction
Edited by Pellicer, E., Adam, J. M., Yepes, V., Singh, A., and Yazdani, S.

ISBN: 978-0-9960437-4-8

A METHOD FOR STATIC ANALYSIS OF CABLE-STAYED STRUCTURES SUBJECTED TO IN-PLANE LOADS

LAI TUONG NGUYEN[1] and THANG BA PHUNG[2]

[1]*Institute of Techniques for Special Engineering, LeQuyDon Technical University, Vietnam*
[2]*Dept of Bridge Engineering, University of Transport Technology, Vietnam*

Present cable theory which formulated from force balance equation of single cable under self-weight and forms a catenary shape of deflection, that is nonlinear; therefore to determine displacement, deformation and tension forces of the cable and cable-stayed structures we need to provide some additional assumptions of cables and use iteration calculation. This paper presents a new method for static analysis of cable-stayed structures subjected to in-plane loading. By combination of the Gaussian Extreme Principle method and virtual displacement principle, authors to formulate and solve nonlinear equation system of cable-stayed structures, which ensured forces balancing as well as continuity of displacements and deformations of structures. This method allows for simultaneous determination of displacement, deformation and internal forces of cable-stayed structures without any other additional hypothesis, which is different from present cable theory.

Keywords: Non-linear analysis, Gaussian extreme principle, Virtual displacement, Numerical analysis.

1 INTRODUCTION

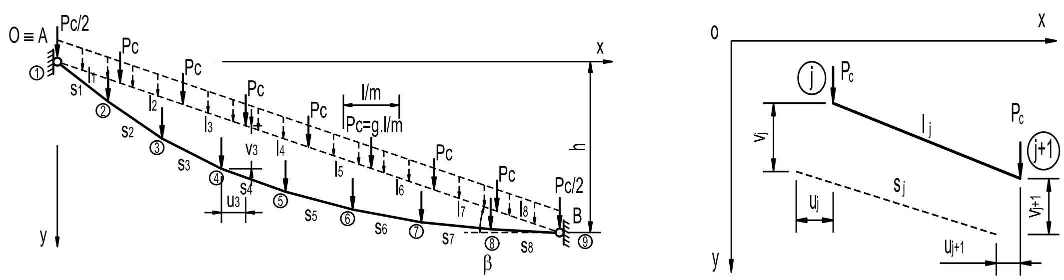
In the analysis of cable-supported structures such as suspension or cable-stayed bridges to date still use the classical theory of flexible cable which forms the deflection shape of catenary or parabola when cable subjected to uniformly distributed load along cable length or over the span. The classical theory of cable formulated from force balance equation of single cable, which is exact and implicit formulation; therefore to determine displacement, deformation and tension forces we need to provide cable dip or horizontal tension force and use iteration calculation as stated in literature (Pugsley 1957) and (Podolny and Scalzi 1986). Due to complexity of cable calculation, in the analysis of cable-stayed bridges simplifications has been made to formulate and solve the equations. Models of tension-only truss member with equivalent modulus (Podolny and Scalzi 1986, Walther *et.al.* 1988, Troisky 1977) or equivalent cross-sectional area (Petropavlovsky 1985) to account for cable sag has been use by researchers as assumptions for linearization in simplified analytical method. In last few decades, researchers have been used basic flexible cable element of catenary shape or parabolic shape to develop finite element solver for cable structures or cable-stayed structures. Attempts were made to formulate stiffness matrix of cable element that satisfy large displacement field of cables during loading of structures.

In this paper, a new method for static analysis of plane problem of cable-stayed structures is presented. Structures are modeled by finite elements with cable described by segmented tension stress elements; columns and beams are described by bending elements. Gaussian Extreme Principle method and principle of virtual displacement are applied to formulate the equations of problem, and then nonlinear programing solves them. This method is well suited for plane static analysis of cable-stayed structures as indicated hereafter, allows for simultaneous determination of displacement, deformation and internal forces of cable-stayed structures without any other additional hypothesis.

2 FORMULATION OF PROBLEM

2.1 Formulation for Single Inclined Cable

Consider a single cable in cable-stayed structures, static loading pattern for cables in these types of structures can be one or combined of the following: self-weight load, concentrated loads, prestressed forces and temperature variation. In calculation, one can divide cable into successive tension-only truss segments, which connected by frictionless pins as shown in Figure 1; distributed self-weight load is converted to concentrated load at connected points of segments.



a) Single cable and equivalent segmented. b) Movement of cable element.

Figure 1. Inclined single cable subjected to self-weight.

In the analysis, we accept the following conditions: (1) cable is completely flexible with uniform cross-sectional area, no bending effect occurred in the cable; (2) cross-sectional area of cable remains constant during deformation; (3) during deformation, stresses only develop in direction normal, and are uniformly distributed, to cross-sectional area.

According to Gaussian Extreme Principle method (Cuong 2005), constraint function of a deformable mechanical structure considers all components – external/internal forces, displacement /strain – of structure. For Figure 1, constraint function can be formulated as:

$$Z = \sum_{i=1}^{m} T_i \varepsilon_i l_i - \sum_{j=1}^{n} P_c v_j \rightarrow \min \tag{1}$$

where T_i, ε_i, l_i respectively are tensile force, elongation strain and undeformed length of i^{th} ($i=1 \div m$) cable segment; P_c, v_j respectively are concentrated load and displacement of cable at point j^{th} ($j=1 \div n=1 \div m-1$); m is number of segment. Tensile force in i^{th} cable segment:

$$T_i = EA.\varepsilon_i = EA\left(s_i - l_i\right)/l_i \tag{2}$$

where s_i is deformed length of i^{th} cable segment, which can be determined as:

$$s_i = \sqrt{\left(x_{i+1}^* - x_i^*\right)^2 + \left(y_{i+1}^* - y_i^*\right)^2}\,;\; x_i^* = x_i + u_i \;;\; x_{i+1}^* = x_{i+1} + u_{i+1} \;;\; y_i^* = y_i + v_i \;;\; y_{i+1}^* = y_{i+1} + v_{i+1} \tag{3}$$

In Eq. (3), u_i, v_i and u_{i+1}, v_{i+1} respectively denote displacement components at node i^{th} and node $(i+1)^{th}$ in horizontal and vertical directions. Condition for minimization of constraint function in Eq. (1) is its variational equal zero, $\delta Z = 0$. According to principle of virtual displacement, while taking variational of Eq. (1) ones consider ε_i independent with T_i, and v_j independent with P_c, therefore:

$$\delta Z = \sum_{i=1}^{m} T_i \delta\varepsilon_i l_i - \sum_{j=1}^{n} P_c \delta v_j = 0 \tag{4}$$

substitute Eq. (2) and (3) into Eq. (4) and taking partial derivative with respect to u_i, v_i, u_{i+1}, v_{i+1}:

$$\frac{\partial Z}{\partial u_i} = \sum_{i=1}^{m} T_i l_i \frac{\partial \varepsilon_i}{\partial u_i} = 0 \ ; \ \frac{\partial Z}{\partial v_i} = \sum_{i=1}^{m} T_i l_i \frac{\partial \varepsilon_i}{\partial v_i} - P_c = 0 \tag{5}$$

Let $l_i = l/m;\ l_{i,x} = l_i \cos\beta_i;\ l_{i,y} = l_i \sin\beta_i;\ P_c = l_i q_c$, to get a system of nonlinear equations for cable with unknowns: u_i, v_i, u_{i+1}, v_{i+1}. These conditions are of force balance at all nodes of deformed cable. Solving these equations for nodal displacements to determine deformed position of cables, utilize Eq. (2) and Eq. (3) to calculate tensile forces in every cable segment.

2.2 Formulation for Beam and Column

Let consider pylon and beam in the cable-stayed structures as deep beam. In bending stage, the rotation of the beam's cross section rotates due to bending moment and shear deformation as:

$$\frac{dw}{dx} = \theta_x + \gamma;\ \gamma = k\frac{Q}{G.A} \tag{6}$$

where θ_x is the rotation due to moment M , γ is the shear deformation due to shear force Q, G is shear modulus of the beam's material, A is the cross sectional area of the beam, k is shear coefficient. From Eq. 6, one can write the rotation due to moment as (Eq. 7):

$$\theta_x = \frac{dy}{dx} - k\frac{Q}{G.A} \tag{7}$$

Let χ is the curvature of the beam's deflection or bending deformation, ones have:

$$\chi = -\frac{d\theta_x}{dx}; M = EJ.\chi = EJ\left(-\frac{d\theta_x}{dx}\right) = EJ\left(-\frac{d^2y}{dx^2} + \frac{d\gamma}{dx}\right) = EJ\left(-\frac{d^2y}{dx^2} + \frac{k}{GA}\frac{dQ}{dx}\right) \tag{8}$$

According to Gaussian Extreme Principle method (Cuong 2005), constraint function for a beam of length l subjected to uniformly distributed load q can be formulated as in Eq. (9) and conditions for minimization of this function is written in Eq. (10) similarly, for cable structure.

$$Z = \int_0^l M\chi dx + \int_0^l Q\gamma dx - \int_0^l qy dx \to \min \tag{9}$$

$$\delta Z = \int_0^l M\delta\chi dx + \int_0^l Q\delta\gamma dx - \int_0^l q\delta y dx = 0 \tag{10}$$

In this study, mixed finite elements with unknown variables are deflections and first derivative of deflections at both ends, shear forces at both ends and at mid-point of elements for bending structure (see Figure 2), were used; therefore, vector of unknown variables for each bending element is $\{R\}_i = [w_1 \quad w_2 \quad \theta_1 \quad \theta_2 \quad Q_1 \quad Q_2 \quad Q_3]_i^T$. Deflection and shear forces of the element can be approximated by polynomial form as:

$$w = N_1(x)w_1 + N_2(x)w_2 + N_3(x)\theta_1 + N_4(x)\theta_2; \quad Q = N_5(x)Q_1 + N_6(x)Q_2 + N_7(x)Q_3 \qquad (11)$$

with the interpolating functions of the element have form as:

$$\begin{aligned} &N_1(x) = f_1 = (x-1)^2(x+2)/4 \text{ ; } N_2(x) = f_2 = (x+1)^2(-x+2)/4 \\ &N_3(x) = f_3 = (x-1)^2(x+1)/4 \text{ ; } N_4(x) = f_4 = (x+1)^2(x-1)/4 \\ &N_5(x) = f_5 = x(x-1)/2 \text{ ; } N_6(x) = f_6 = (1-x)(1+x) \text{ ; } N_7(x) = f_7 = x(1+x)/2 \end{aligned} \qquad (12)$$

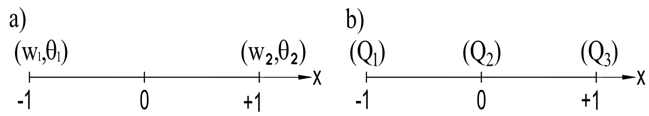


Figure 2. Mixed finite element for bending structure with shear deformation.

Denote $a^{'} = da/dx$; $a^{''} = d^2a/dx^2$, and we can get the matrix form of w and Q, including their derivatives easily as:

$$w = [w]\{R\} = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad 0 \quad 0 \quad 0]\{X\}; \quad Q = [Q]\{R\} = [0 \quad 0 \quad 0 \quad 0 \quad f_5 \quad f_6 \quad f_7]\{R\} \qquad (13)$$

$$\begin{aligned} &w^{''} = d^2w/dx^2 = \left[w^{''}\right]\{R\} = \left[f_1^{''} \quad f_2^{''} \quad f_3^{''} \quad f_4^{''} \quad 0 \quad 0 \quad 0\right]\{R\} \\ &Q^{'} = \left[Q^{'}\right]\{R\} = \left[0 \quad 0 \quad 0 \quad 0 \quad f_5^{'} \quad f_6^{'} \quad f_7^{'}\right]\{R\} \end{aligned} \qquad (14)$$

Substitute Eq. (13) and Eq. (14) into Eq. (6) and Eq. (8), ones get matrices of moment, rotation due to shear forces and rotation due to moment of an element as:

$$M = EJ\chi = EJ[\chi]\{R\} \text{ ; } \gamma = \frac{kQ}{GA} = \frac{k}{GA}[Q]\{R\} \text{ ; } \chi = [\chi]\{R\} = \left(-\left[w^{''}\right] + \frac{k}{GA}[Q']\right)\{R\} \qquad (15)$$

Consider a bending element with length of Δx, from Figure 2 and account for above matrices, after some transformation lead to matrix form of equation for element:

$$\frac{\partial Z}{\partial R_i} = 0; \ (i = 1 \div 7) \quad \Leftrightarrow \quad \frac{\Delta x}{2}\left(EJ\int_{-1}^{1}[\chi][\chi].dx + \frac{k}{GF}\int_{-1}^{1}[Q][Q].dx\right)\{u\} = \{P\}[w] \qquad (16)$$

Set $[A_e] = \frac{\Delta x}{2}\left(EJ\int_{-1}^{1}[\chi][\chi].dx + \frac{k}{GA}\int_{-1}^{1}[Q][Q].dx\right)$, Eq. (16) can be rewritten as Eq.(17):

$$[A_e]\{R\} = \{P\}[w] \ \Rightarrow \ \{R\} = \{P\}[w][A_e]^{-1} \qquad (17)$$

2.3 Formulation for Static Problem of Plane Cable-Stayed Structures

Consider a typical simple cable-stayed structure as shown in Figure 3. In the deformed stage of structure, pylon and beam only take bending and shear deformation, no axial deformation occurred as their axial stiffness are very large, therefore they only take deflection in the direction normal to longitudinal axis.

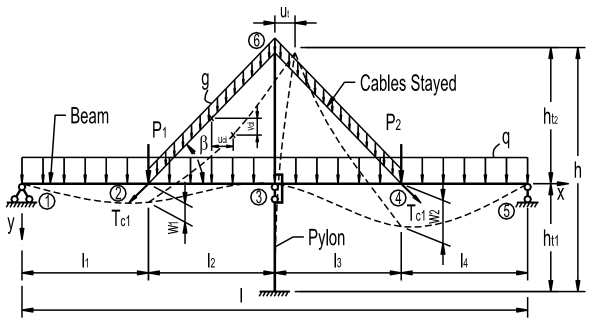


Figure 3. Plane cable-stayed structure.

To formulate equations of the structure, subdivide it into number of finite elements, convert all distributed loads to equivalent concentrated loads at end points of finite element. With stage of displacements and deformation of structure mentioned above, ones get constraint function of structure as:

$$\begin{aligned} Z = & \underbrace{\sum_{i=1}^{n_b}\int_0^{l_i} M_i\chi_i dx + \sum_{i=1}^{n_b}\int_0^{l_i} Q_i\gamma_i dx - \sum_{i=1}^{n_b}\left(P_{1i}w_{1i} + P_{2i}w_{2i}\right)}_{beam} + \\ & + \underbrace{\sum_{j=1}^{n_p}\int_0^{l_j} M_j\chi_j dx + \sum_{j=1}^{n_p}\int_0^{l_j} Q_j\gamma_j dx - \sum_{j=1}^{n_p}\left(P_{1j}w_{1j} + P_{2j}w_{2j}\right)}_{pylon} + \underbrace{\sum_{i=1}^{n}\left(\sum_{j=1}^{m} T_{ij}\varepsilon_{ij}l_{ij} - \sum_{j=1}^{m-1} P_c v_{ij}\right)}_{cables} \rightarrow \min \end{aligned} \tag{18}$$

where n_b, n_p, n, m are number of beam elements, pylon elements, cables and segments on each cable respectively. Continuity conditions between cables, beam and pylon at connected points must be satisfied for displacements and forces. Condition for minimization of constraint function in Eq. (18) is its variational equal zero, $\delta Z = 0$; by taking variational operator with respect to unknown variables ($\{R\}$, u_{ij}, v_{ij}), ones get system of nonlinear equations as (Eq. 19):

$$\frac{\partial Z}{\partial R_i} = 0\ \left(i = 1 \div 7\right); \frac{\partial Z}{\partial \mathrm{u}_{ij}} = \frac{\partial}{\partial \mathrm{u}_{ij}}\sum_{i=1}^{n}\left(\sum_{j=1}^{m} T_{ij}\varepsilon_{ij}l_{ij} - \sum_{j=1}^{m-1} P_c v_{ij}\right) = 0;\ \frac{\partial Z}{\partial \mathrm{v}_{ij}} = \frac{\partial}{\partial \mathrm{v}_{ij}}\sum_{i=1}^{n}\left(\sum_{j=1}^{m} T_{ij}\varepsilon_{ij}l_{ij} - \sum_{j=1}^{m-1} P_c v_{ij}\right) = 0 \tag{19}$$

2.4 Formulation and Solution of Nonlinear Programing Problem for Structure

Plane static problem of cable-stayed structure lead to system of nonlinear equations as stated by Eq. (18), these equations can be solved by different methods. In this study, authors use nonlinear programing method to solve the formulated equations as the constraintts of problem consisted of equality and inequality. The objective function for problem is to minimize total strain energy of all cables in the structure. The problem can be restated as to minimize $\sum_{i=1}^{n}\int_0^{l_d}\frac{T_{ij}^2}{2\mathrm{EA}}ds = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}\frac{T_{ij}^2 l_{ij}}{2\mathrm{EA}}\right)$ subjected to: linear and nonlinear equality expressed by Eq. (18); nonlinear inequalities $T_{ij} > 0\ \left(i = 1 \div n;\ j = 1 \div m\right)$. A computer code was written in MATLAB to solve the stated problem with assistant from *fmincon* function of Optimization Toolbox.

3 NUMERICAL EXAMPLE

Consider a continuous double-span cable-stayed structure as shown in Figure 4. Analysis with consideration of shear deformation were carried out by the computer code mentioned above, results are shown graphically in the Figure 4. All force-balanced check is satisfied. Numerical results from presented example in this study have shown the correctness of the proposed method as well as accuracy of the procedure and program made by authors.

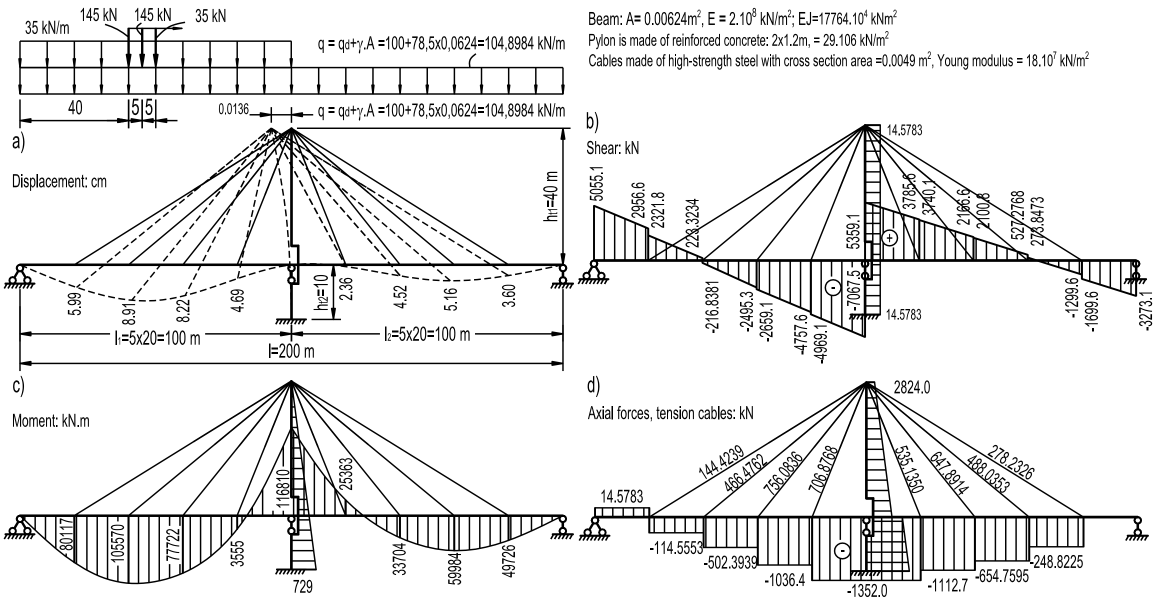


Figure 4. Cable-stayed structure in numerical example.

4 CONCLUSIONS

This paper briefly presents a new method for static analysis of cable-stayed structure subjected to in plane loading. This method applies the Gaussian Extreme Principle method to formulate the equations of structure and solve the formulated problem by nonlinear programing. The advance of this method is formulation can be made simply straight forward. The proposed method allows for static analysis of cable-stayed structure according to plane scheme with consideration of geometrical nonlinearity of cables as well as shear deformation effect in the bending element; additionally it allow for simultaneous determination of displacement and internal forces of structure without too much hypothesis as in traditional methods.

References

Cuong, H. H., Gaussian Extreme Principle Method, *Journal of Science and Techniques*, Le Quy Don University, No. 4, December 2005.

Podolny Jr., W. and Scalzi, J. B., *Construction and Design of Cable-Stayed Bridges*, USA, 1986.

Petropavlovsky, A. A., *Cable-stayed Bridges*, Moscow, 1985. (Translated from: А.А. Петропавловский, *Вантовые Мосты*, Москва, 1985).

Pugsley, Sir A., *The theory of suspension bridges*, Edward Arnold Ltd., London, 1957.

Troisky M. S., *Cable-stayed Bridges: Theory and Design*, Rainbom-Bridge Book Co., 1977.

Walther, R., Houriet, B., Isler, W., and Moïa, P., *Cable stayed bridges*, Thomas Telfford Ltd., London, 1988.