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THE FOUNDATIONS OF CONTINUUM DAMAGE MECHANICS

SIAMAK YAZDANI[1], SEVENN BORGERSEN[2], ASLI PELIN GURGUN[3], and HOSSEIN NAZARI[4]

[1]*Dept of Civil Engineering, California State University, Pomona, USA*
[2]*SEB Corp, Eagan, Minnesota, USA*
[3]*Dept of Civil Engineering, Yildiz Technical University, Istanbul, Turkey*
[4]*Dept of Mechanical Engineering, NDSU, Fargo, USA*

Damage Mechanics has become a useful theory in describing the nonlinear behavior of solids driven by the nucleation and growth of cracks and microcracks. This approach, based on the first principles of mechanics and thermodynamics, has also been combined with classical theories of plasticity to address a wide range of loading applications. In spite of the many different damage mechanics models and representations that are proposed, the foundation of damage mechanics is not well understood or at least not thoroughly published giving rise to the many inaccurate definitions and formulations. The intent of this paper is to provide the background of the continuum damage mechanics outlining the fundamentals on which this field theory is set up. The internal variable theory of continuum thermodynamics is reviewed and is shown that with Legendre transformation technique, various potential functions can be developed for damage mechanics formulation in either stress or strain space. The concept of constrained or neighboring equilibrium state is also introduced and is explained. The paper will conclude with the derivation of the general damage potential and a suggestion is given for the isotropic damage formulation with the resulting uniaxial stress-strain relation.

Keywords: Response tensor, Plasticity, Dissipation inequality, Isothermal deformations.

1 INTRODUCTION

The observed nonlinear behavior in brittle solids stems from two main meso-structural changes within the materials. One is the development and propagation of microcracks and micro-voids. The process of cracking destroys material bonds and leads to the reduction in the material stiffness. Failure occurs where these microcracks link up and form a major crack and fault line. This behavioral pattern occurs mostly when zero or low confining pressure exists. Uniaxial tension, uniaxial compression, biaxial compression and tension are typical load paths under which microcracks propagation and coalescence dominate solid behavior.

When the lateral confining pressure is large, the formation of microcracks gets inhibited and/or delayed. The classical experimental work of Hueckel and Maier (1977) on rocks has shown that under sufficiently large confining pressure permanent deformations due to plastic flow and void closures occur and that these meso-structural changes, as the other category of internal changes, lead to no alteration of material stiffness. To model such diverse behavior observed in solid materials, different classes of theories have been proposed.

There are three levels of damage mechanics theories that one may employ to model nonlinear behavior of brittle solids. One is the micro-level modeling incorporating material science approach where the formulation is intended to correlate closely with the physics of micro-structural changes. Although, this approach is fundamental, its usefulness is rather limited in engineering applications where complex loading paths, such as proportional and non-proportional paths, load reversal, etc, must be considered. The next level used by researchers is referred to as the meso-level modeling where microcrack development and kinetics at a smaller scale than a continuum is utilized. This approach is interesting and promising involving discrete field theories such as fracture mechanics to model microcrack initiation and propagation in the medium and at the interfaces. The utility of this approach is restricted to simple load paths and the end an averaging or smearing formalism is used to predict macro behavior; hence losing the fine details used up to that point. For most engineering applications and modeling the third approach, namely the macro-level modeling (Saboori *et al*. 2014, 2015) is overwhelmingly used and appears in numerical simulation codes such as ANSYS and ABAQUS.

This intent of this paper is therefore to present a continuum level class of damage mechanics theories that is based on the first principles of mechanics and thermodynamics. Some fundamental concepts critical to the understanding of the theory is explained in the body of the paper.

2 FORMULATION

Central to the continuum damage mechanics is the concept of neighboring equilibrium state (Ortiz 1985, Yazdani and Karnawat 1996, Yazdani 1993). This can be explained further in a schematic representation of a crack growth in Figure 1. From fracture mechanics, we know that an active and stable crack of length a would reach another equilibrium state of length b once acted upon a sufficiently large load that overcomes local fracture resistance.

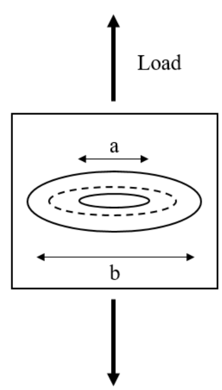

Figure 1. Schematic representation of crack growth.

The concept of the neighboring equilibrium state would allow for the mathematical formulation of crack growths at lengths a and b, but also for any assumed length in between; and hence the term neighboring equilibrium state is used. Assuming that the material internal energy under stress exists, the internal variable theory of thermodynamics (Lubliner 1972) can be used to cast an equilibrium state in terms of the Helmholtz Free Energy (HFE), $A(\boldsymbol{\epsilon}, k)$, or the Gibbs Free

Energy (GFE), $G(\boldsymbol{\sigma}, k)$ where $\boldsymbol{\epsilon}$ is the strain tensor, $\boldsymbol{k}$ represents a cumulative damage parameter, and $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor. The thermodynamics functions HFE and GFE are related to each other through an elegant mathematical transformation technique known as "Legendre" transformation.

When formulation is desired to be developed in terms of the strain components, then the HFE function is used. When stress space formulation is desired, the GFE function is utilized. For isothermal deformations, as assumed here, the internal dissipation inequality as a consequence of the second law of thermodynamics, can be expressed either in terms of the HFE as shown in Eq. (1) and Eq. (2):

$$\dot{A} + \boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}} \geq 0 \tag{1}$$

or in terms of GFE as:

$$\dot{\boldsymbol{G}} - \boldsymbol{\epsilon} : \dot{\boldsymbol{\sigma}} \geq \boldsymbol{0} \tag{2}$$

where the "super dots" denote the time rate. For the rest of this paper the foundation of the damage mechanics will be presented in the stress space using GFE.

Cracking and microcracking in brittle solids is known to destroy material bonds and thus affecting the compliance of the solid. Denoting the compliance tensor as $\boldsymbol{C}(k)$ and the initial compliance of the undamaged state as $\boldsymbol{C^0}$ is then shown in Eq. (3):

$$\boldsymbol{C}(\boldsymbol{k}) = \boldsymbol{C^0} + \boldsymbol{C^c}(\boldsymbol{k}) \tag{3}$$

where $\boldsymbol{C^c}$ is the added compliance tensor reflecting the effects of damage. The total deformation can also be shown in terms of the components as Eq. (4):

$$\begin{aligned} \boldsymbol{\epsilon} &= \boldsymbol{\epsilon^0} + \boldsymbol{\epsilon^D} + \boldsymbol{\epsilon^r}(k) \\ &= \boldsymbol{C^0}:\boldsymbol{\sigma} + \boldsymbol{C}^c(k):\boldsymbol{\sigma} + \boldsymbol{\epsilon^r}(k) \end{aligned} \tag{4}$$

where $\boldsymbol{\epsilon^0} = \boldsymbol{C}^0:\boldsymbol{\sigma}$ is the elastic component of the deformation, $\boldsymbol{\epsilon}^D(k) = \boldsymbol{C^c}(k):\boldsymbol{\sigma}$ is referred to as the elastic damage deformation, and $\boldsymbol{\epsilon^r}(k)$ is the inelastic or un-recoverable/permanent deformation caused by the misfit of crack surfaces. The schematic representation of these components is shown in Figure 2 for a uniaxial stress path.

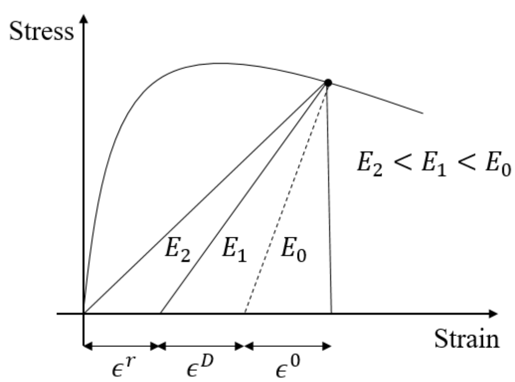

Figure 2. Schematic representation of deformation components.

Eq. (2) leads to two fundamental equations that:

$$\boldsymbol{\epsilon} = \frac{\partial G}{\partial \boldsymbol{\sigma}} \tag{5}$$

and

$$\frac{\partial G}{\partial k}\dot{k} \geq 0 \tag{6}$$

where, Eq. (5) states that the GFE is a potential to the strain tensor and where Eq. (6) is called the dissipation inequality. Integrating Eq. (5) and using the decomposition structure shown in Eq. (4), a general form of the GFE as obtained as:

$$G(\boldsymbol{\sigma}, k) = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\sigma}{:}\,\boldsymbol{C}{:}\,\boldsymbol{\sigma} + \boldsymbol{\epsilon}^{\boldsymbol{r}}(k){:}\,\boldsymbol{\sigma} - A^i(k) \tag{7}$$

where $A^i(k)$ is the inelastic component of the HFE associated with the surface tension of the crack surfaces and arises as the constant of the integration. Incorporating Eq. (7) into Eq. (6) yields:

$$\frac{\partial G}{\partial k} = \frac{1}{2}{:}\,\boldsymbol{\sigma}{:}\frac{\partial \boldsymbol{C}^{\boldsymbol{c}}}{\partial k}{:}\,\boldsymbol{\sigma} + \frac{\partial \boldsymbol{\epsilon}^{\boldsymbol{r}}(k)}{\partial k}{:}\,\boldsymbol{\sigma} + \frac{\partial A^i(k)}{\partial k} \geq 0 \tag{8}$$

where it is assumed that damage is irreversible; i.e. $\dot{k} \geq 0$.

Utilizing Eq. (8), we can now obtain a damage potential $\boldsymbol{\psi}(\boldsymbol{\sigma}, k)$ by introducing a positive function $\boldsymbol{g}^{\mathbf{2}}(\boldsymbol{\sigma}, k)$ as shown in Eq. (9) such that:

$$\psi(\boldsymbol{\sigma}, k) = \frac{1}{2}\boldsymbol{\sigma}{:}\frac{\partial \boldsymbol{C}^c}{\partial k}{:}\,\boldsymbol{\sigma} + \frac{\partial \boldsymbol{\epsilon}^r}{\partial k}{:}\,\boldsymbol{\sigma} + \frac{\partial A^i(k)}{\partial k} - g^2(\boldsymbol{\sigma}, k)$$

$$= \frac{1}{2}\boldsymbol{\sigma}{:}\frac{\partial \boldsymbol{C}^{\boldsymbol{c}}}{\partial k}{:}\,\boldsymbol{\sigma} + \frac{\partial \boldsymbol{\epsilon}^{\boldsymbol{r}}}{\partial k}{:}\,\boldsymbol{\sigma} - \frac{1}{2}t^2(\boldsymbol{\sigma}, k) = 0 \tag{9}$$

where $t^2(\boldsymbol{\sigma}, k) = -2\frac{\partial A^i}{\partial k} + 2g^2(\boldsymbol{\sigma}, k)$ is regarded as the damage function. We note that the identification of either of the functions $\frac{\partial A^i}{\partial k}$ or $g^2(\boldsymbol{\sigma}, k)$ is not necessary as long as the damage function $t(\boldsymbol{\sigma}, k)$ is obtained through experiments. The final general form of the damage surface, $\psi(\boldsymbol{\sigma}, k)$ can then be stated by introducing a damage response tensor $\boldsymbol{R}$ and a flow-rule for $\dot{\boldsymbol{\epsilon}}^{\boldsymbol{r}}$ (Eq. (10) and Eq. (11)) with **M** being identified as the second order inelastic flow tensor, such that:

$$\dot{\boldsymbol{C}}^{\boldsymbol{c}}(k) = \dot{k}\boldsymbol{R} \tag{10}$$

$$\dot{\boldsymbol{\epsilon}}^{\boldsymbol{r}}(k) = \dot{k}\boldsymbol{M} \tag{11}$$

Introducing these into the structure of Eq. (9) yields the general form of the damage surface as Eq. (12):

$$\psi(\boldsymbol{\sigma}, k) = \frac{1}{2}\boldsymbol{\sigma}{:}\,\boldsymbol{R}{:}\,\boldsymbol{\sigma} + \boldsymbol{M}{:}\,\boldsymbol{\sigma} - \frac{1}{2}t^2(\boldsymbol{\sigma}, k) = 0 \tag{12}$$

Different tensors of $\boldsymbol{R}$ and $\boldsymbol{M}$ define different models that are published in the literature. If $\boldsymbol{R}$ is postulated as an isotropic tensor, the predicted results would be considered isotropic. This is not usually observed in brittle solids, but under blast loading some isotropic and uniform distribution of damage has been reported, making isotropic from of $\boldsymbol{R}$ a viable choice in some cases.

3 EXAMPLE – UNIAXIAL LOAD PATH

With $\boldsymbol{R}$ speculated to be proportional to an isotropic tensor, and with $\boldsymbol{M}$ being set as a null tensor for simplicity, the uniaxial stress can be reduced from Eq. (12) to be (Eq. 13)):

$$\sigma_1 = t(k) \tag{13}$$

which implies that the damage function could be experimentally determined from a simple uniaxial test. When the tensor M is set to be a null tensor, no inelastic damage strain, $\boldsymbol{\epsilon}^{\boldsymbol{r}}(k)$, is predicted and the response is classified as being elastic-perfectly brittle. The associated uniaxial strain, $\boldsymbol{\epsilon_1}$ then obtained by integrating the rate form of the Eq. (4), in general or for the case of simply load path as considered here simply from Eq. (4) and Eq. (13) as Eq. (14):

$$\epsilon_1 = \left(\frac{1}{E_0} + k\right)\sigma_1 \tag{14}$$

A normalized stress-strain curve for using the damage function by Ortiz (1985) is shown in Figure 3, where $\boldsymbol{f_t}$ is the uniaxial tensile strength of solid and $\boldsymbol{\epsilon}_t$ is the corresponding strain.

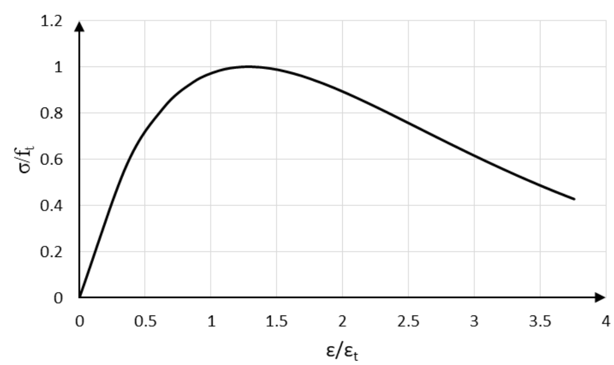

Figure 3. Normalized stress-strain curve of a solid under uniaxial tensile loading.

4 CONCLUSIONS

Damage mechanics is a viable field theory for solids where the development of cracks and microcracks render the material damaged and hence more compliant. A class of damage mechanic theories is outlined in this paper defining all critical assumptions and terms not usually covered in papers on damage mechanics. This approach was shown to be based on the first principles of mechanics and thermodynamics utilizing the second law to obtain dissipation inequality leading to the formulation of a damage potential. In this way, the number of material parameters is usually small, and pertain to the physical attributes of the macro behavior such as elasticity, Poisson's ratio, bi-axial strength ratios, etc. The general form of the damage surface was derived and damage response tensors were defined. An example of stress-strain relation was provided for a special case of isotropic tensor for $\boldsymbol{R}$, making the interpretation of the damage parameter rather straight forward and easy to understand.

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