AN ITERATIVE HDMR APPROACH FOR ENGINEERING RELIABILITY ANALYSIS

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Engineering reliability analysis has long been an active research area. Surrogate models, or metamodels, are approximate models that can be created to replace implicit performance functions in the probabilistic analysis of engineering systems. Traditional 1st-order or second-order high dimensional model representation (HDMR) methods are shown to construct accurate surrogate models of response functions in an engineering reliability analysis. Although very efficient and easy to implement, 1st-order HDMR models may not be accurate, since the cross-effects of variables are neglected. Second-order HDMR models are more accurate; however they are more complicated to implement. Moreover, they require much more sample points, i.e., finite element (FE) simulations, if FE analyses are employed to compute values of a performance function.

In this work, a new probabilistic analysis approach combining iterative HDMR and a first-order reliability method (FORM) is investigated. Once a performance function is replaced by a 1st-order HDMR model, an alternate FORM is applied. In order to include higher-order contributions, additional sample points are generated and HDMR models are updated, before FORM is reapplied. The analysis iteration continues until the reliability index converges. The novelty of the proposed iterative strategy is that it greatly improves the efficiency of the numerical algorithm. As numerical examples, two engineering problems are studied and reliability analyses are performed. Reliability indices are obtained within a few iterations, and they are found to have a good accuracy. The proposed method using iterative HDMR and FORM provides a useful tool for practical engineering applications.

Keywords: High dimensional model representation (HDMR), Probabilistic analysis, Engineering applications, Surrogate model, First-order reliability method (FORM).

1 INTRODUCTION

Reliability analysis of practical engineering applications relies on sampling techniques and gradient-based methods. Sampling methods are very popular (Rubinstein 1981, Au and Wang 2014) that can be applied and coupled with FE analyses in a straightforward manner, since derivative or sensitivity calculations are not required. For problems that require time-consuming response simulations, these methods may not be efficient or feasible. First- and second-order reliability methods (FORM/SORM) can also be applied in the reliability analysis of engineering systems (Hasofer and Lind 1974, Hohenbichler et al. 1987). However, they also have to be integrated with FE analysis software, which may not be easy especially if analytical derivatives are provided.

In this work, a new approximation model is proposed and applied in engineering reliability analysis. This belongs to a broad class of methods, referred to as surrogate models (Bai et al. 2012). In the literature, different surrogate models have been developed, including simple
quadranic functions and some more complicated functions (Kim and Na 1997, Fang and Wang 2008, Wang et al. 2020). HDMR framework has been developed to replace a function with component functions of various orders (Li et al. 2001, Wu et al. 2016). HDMR has shown to be efficient, but traditional 1st-order or second-order HDMR methods have their limitations. It is useful to investigate an iterative HDMR method that can have advantages of both the 1st-order and second-order HDMR models. The iterative approach combines a HDMR framework and FORM, so the reliability analysis is performed in a repeated manner. In each iteration, an explicit HDMR model of the performance function is constructed. The component functions of HDMR models are expressed using augmented radial basis functions (RBFs) (Hassing et al. 2010, Yin et al. 2016). Reliability index and failure probability are computed using FORM, once explicit RBF-HDMR surrogate models are available. The model accuracy of RBF-HDMR is improved during iterations, and a convergence of the reliability index can be obtained.

In Section 3 of the paper, the proposed method is first introduced. The reliability analysis approach includes the HDMR framework, the RBF-HDMR method, FORM, and an iterative technique. To demonstrate the proposed approach, two civil engineering problems are solved in Section 4. Section 5 gives a summary of the work.

2 RELIABILITY ANALYSIS

A reliability analysis of an engineering system or component is to calculate the following multi-dimensional integration, as

\[ P_f \equiv P(g(\mathbf{x}) \leq 0) = \int_{g(\mathbf{x}) \leq 0} p_x(\mathbf{x}) d\mathbf{x} \quad (1) \]

where \( P_f \) is the failure probability, \( g(\mathbf{x}) \) is a performance function in terms of the random variable vector, \( \mathbf{x} \), and \( p_x(\mathbf{x}) \) is the probability density function Eq. (1).

3 THE PROPOSED RELIABILITY ANALYSIS APPROACH

A brief introduction of the HDMR framework, RBF-HDMR method, FORM, and the proposed iterative approach is given in this section.

3.1 The HDMR Framework

The HDMR of any multivariable function, \( g(\mathbf{x}) \), can be written as in Eq. (2) (Chowdhury et al. 2009)

\[ g(\mathbf{x}) = g_0 + \sum_{i=1}^{m} g_i(\mathbf{x}_i) + \sum_{1 \leq i_1 < i_2 \leq m} g_{i_1 i_2}(x_{i_1}, x_{i_2}) + \sum_{1 \leq i_1 < i_2 < i_3 \leq m} g_{i_1 i_2 i_3}(x_{i_1}, x_{i_2}, x_{i_3}) + \cdots \quad (2) \]

where the constant term \( g_0 \) is the zeroth-order HDMR component function, and it is the value of the original function \( g(\mathbf{x}) \) at the point of expansion, \( \mathbf{x} = \mathbf{c} \), as shown in Eq. (3)

\[ g_0 = g(\mathbf{c}) \quad (3) \]

In Eq. (2), \( g_i(\mathbf{x}_i) \) is a function of only one variable, \( x_{i} \), as shown in Eq. (4)

\[ g_i(x_i) = g(x_i, \mathbf{c}^i) - g_0 \quad (4) \]

which is a 1st-order HDMR component function. It is important to note that \( g_i(x_i) \) can be linear or nonlinear, and as shown in Eq. (5)

\[ g(x_i, \mathbf{c}^i) = g(c_{i_1}, ..., c_{i-1}, x_i, c_{i+1}, ..., c_m) \quad (5) \]
A HDMR model written in Eq. (2) is indeed a combination of component functions, and can be further simplified as shown in Eq. (6):

\[ g(\mathbf{x}) = g_0 + \sum_{i=1}^{m} g_i(x_i) + \mathbb{R}_2 \]  

(6)

where \( \mathbb{R}_2 \) is a function that includes the contribution from higher-order component functions. If \( \mathbb{R}_2 \) is neglected, Eq. (6) becomes a 1st-order HDMR surrogate model, which only includes the lower-order component functions.

### 3.2 The RBF-HDMR Method

To explicitly express a HDMR model in Eq. (6), an approximate method can be adopted to express the HDMR component functions. In this study, the RBFs are used (Fang and Wang 2008, Hassing et al. 2010). Substitute an RBF of \( g_i(x_i) \) into Eq. (6), an RBF-HDMR surrogate model can be obtained, as

\[ g(\mathbf{x}) = g(\mathbf{c}) + \sum_{i=1}^{m} \left[ \sum_{j=1}^{n_i} \lambda_{ij} \phi\left( \left\| x_i - x_i^j \right\| \right) \right] + \mathbb{R}_2 \]  

(7)

where \( n_i \) is the number of samples used to create \( g_i(x_i) \). To express the residual function, \( \mathbb{R}_2 \), using an RBF, we have

\[ \mathbb{R}_2 = \sum_{j=1}^{n_R} \lambda_{Rj} \phi\left( \left\| \mathbf{x} - \mathbf{x}_j \right\| \right) \]  

(8)

where \( n_R \) is the number of samples used to create \( \mathbb{R}_2 \) Eq. (7).

### 3.3 Form

In an alternate implementation of FORM, based on an expanding ellipsoid concept, the reliability index is written as (Low and Tang 2007, Wang and Fang 2018):

\[ \beta = \min_{\mathbf{x} \in F} \sqrt{\mathbf{n}^T \mathbf{R}^{-1} \mathbf{n}} \]  

(9)

In Eq. (9), the element of vector \( \mathbf{n} \) is

\[ n_i = \frac{x_i - \mu_i}{\sigma_i} = \Phi^{-1}[F(x_i)] \]  

(10)

where \( \Phi^{-1} \) is the inverse cumulative distribution function of a standard normal variable Eq. (10). In order to calculate the reliability index, it is common to apply a nonlinear programming algorithm with \( \mathbf{n} \) treated as the design variable vector (Zhao et al. 2014).

### 3.4 An Iterative Approach

To enhance the model accuracy of traditional HDMR methods, an iterative approach is proposed. The basic idea is to add additional sample points in the neighborhood of the current design point. The method starts with the reliability analysis using a simple and efficient 1st-order RBF-HDMR model of performance function. This only includes the lower-order HDMR component functions in Eq. (7). The design point is found and additional sample points are located. In each of the subsequent iteration, the residual function \( \mathbb{R}_2 \) in Eq. (8) is updated, using all additional sample points. In the examples of this study, the additional sample size is selected to be twice of the size of the random variable vector in each iteration.
4 NUMERICAL EXAMPLES

Two civil engineering problems are studied and their results are presented in this section. One example has a close-form performance function and the other example uses FE analyses to evaluate the implicit performance function.

4.1 A Tunnel Engineering Problem

The first example is a tunnel engineering problem with close-form functions. The radius of a circular tunnel is $r_0 = 1$ m, as shown in Figure 1. The tunnel is subject to the a far field stress, $p_0$, and an internal support pressure, $p_i$. The performance function is given in Eq. (11) and the detailed derivations of the closed-form functions are available (Li and Low 2010, Wang et al. 2016).

$$g(x) = 3 - \frac{r_p}{r_0}$$  \hspace{1cm} (11)

![Figure 1. A circular tunnel.](image)

In this example, Young’s modulus $E$, cohesion $c$, and friction angle $\phi$ are random variables. Moreover, $c$ and $\phi$ are assumed to be negatively correlated (correlation coefficient = $-0.5$) (Zhao et al. 2014). The performance of the tunnel is studied with different far field stresses ($p_0 = 1.5 - 3.0$ MPa) but no support pressure ($p_i = 0.0$). The reliability index $\beta$ is calculated and given in Table 1, along with other reliability analysis results. Figure 2 shows the variations of $\beta$ and failure probability, $P_f$. It is expected that as the $p_0$ increases, the reliability index $\beta$ decreases, and $P_f$ increases. To compare results, the reliability analyses are also performed using closed-form performance function and FORM. Table 1 and Figure. 2 show that the failure probabilities and reliability indices calculated using FORM with and without surrogate models are in close agreement. The probabilistic analysis approach based on FORM and iterative RBF-HDMR method works very well. Only three iterations are required for a converged reliability analysis.

<table>
<thead>
<tr>
<th>Far field stress $p_0$ (Mpa)</th>
<th>Original function</th>
<th>RBF-HDMR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Reliability index $\beta$</td>
<td>1.8777</td>
<td>1.3024</td>
</tr>
<tr>
<td>$E$ (Mpa)</td>
<td>373</td>
<td>373</td>
</tr>
<tr>
<td>Design point $c$ (Mpa)</td>
<td>0.1045</td>
<td>0.1444</td>
</tr>
<tr>
<td>$\phi$ (Degree)</td>
<td>23.6660</td>
<td>23.2959</td>
</tr>
</tbody>
</table>

Table 1. Reliability analysis results.
4.2 A Slope Stability Problem

This is an existing slope stability problem in the literature (Tan and Wang 2009). There are two layers of undrained clay in the slope. To analyze the slope and perform numerical reliability analysis, a two-dimensional FE model is built (Griffiths and Lane 1999). Figure 3 shows the geometry and mesh of the FE model.

![Slope stability problem](image)

Figure 3. A slope stability problem.

The following parameters are adopted for both soil layers in the model: unit weight $\gamma=20$ kN/m$^3$, Young’s modulus $E=1.0E+05$ kPa, Poisson’s ratio $\mu=0.3$, effective friction angle $\phi=0$, and dilation angle $\psi=0$. The independent random variables are the effective cohesion, $c_1$ and $c_2$, of the two layers (Tan and Wang 2009). Both variables follow normal distributions: $c_1$ has a mean value of 50.0 kPa and $c_2$ has a mean value is 73.1 kPa. The coefficient of variation of both variables is 0.25. The slope stability performance function is written as

$$g(x) = \text{F.S.} - 1.0$$

(12)

where F.S. is the factor of safety. To start the reliability analysis, an initial 1st-order RBF-HDMR surrogate model is created using nine samples. In each subsequent reliability analysis iteration, four additional samples are generated. At the convergence of the iterative RBF-HDMR method, five iterations are required. The proposed method works well for this example and a total of $9+4\times4=25$ sample points are generated. The final estimated failure probabilities is 0.0107 and the reliability index is 2.30.

5 SUMMARY AND CONCLUDING REMARKS

An efficient iterative reliability analysis approach is developed and applied to two civil engineering problems. The proposed approach is based on an iterative RBF-HDMR surrogate modeling method. A FORM algorithm is implemented to find the reliability index in a dynamic manner, so that additional sample points are located and the residual function (higher-order
contributions) is constructed. More accurate HDMR models are generated in order to enhance the model accuracy in an iterative manner. The proposed approach works well and only a few iterations and a small number of samples are required to obtain convergence. The iterative RBF-HDMR is useful for practical problems requiring time-consuming FE analyses.

References


