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STABILITY CONSIDERATIONS FOR ELASTIC-PLASTIC PROCESSES WITH DAMAGE

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Classical theory of plasticity is fairly complete with flow rules, convexity of yield surfaces, extremum principles, and the uniqueness theorem. For the strain-hardening plasticity, Drucker's postulates are established and proven based on the plastic-work and energy principles. Plasticity models have been further applied to heterogeneous and cementitious materials with certain degrees of success. In this paper, the stability statements of strain-hardening and strain-softening processes in concrete are examined by utilizing thermodynamic potential functions in the stress space and by applying Euler's theorem of homogenous functions. It is shown that by specifying a strain-hardening parameter to account for the plastic strains and a damage parameter to represent the effect of microcracking, the dissipation inequality can be used to establish the Drucker's stability postulate for the plastic flow within the framework of the internal variable theory of thermodynamics. Using the same approach and assuming uncoupling between plastic flow and microcracking, the formation leads to a softening stability statement for damage processes in concrete.

Keywords: Drucker's postulates, Dissipation, Thermodynamic, Irreversible, Softening.

1 INTRODUCTION

Classical theories of plasticity have been well established. In these theories, the general premise considers a yield surface encompassing an elastic domain. The total strain increment is then additively decomposed into elastic and plastic strain components. Flow rules are subsequently postulated for the plastic strain increments and the necessary and sufficient conditions are stated usually in the Kuhn-Tucker forms. The Drucker's plasticity postulate states that for a stable process in an elastic-plastic deformation, the work done by an external agency on the changes in the displacements it produces must be non-negative (Lubliner 2006). Mathematically, it can be written as shown in Eq. (1).

$$\dot{\varepsilon}^p : \dot{\sigma} \;\; \geq 0 \tag{1}$$

where, $\dot{\varepsilon}^p$ denotes the changes in plastic strain tensor and where $\dot{\sigma}$ represents the changes in the Cauchy stress tensor caused by an external load. The Drucker's postulate leads to the conclusions that the yield surface is convex in the stress space and that the increment of plastic strain tensor is normal to the yield surface. This is referred to as associated flow-rule. The non-negative scalar multiplier in the associated flow-rule is determined from the consistency condition of the yield surface (Lubliner 2006, Neto *et al*. 2008).

In this context, the von Mises yield criterion is most widely used with strain hardening functions and parameters. The von Mises criterion is a pressure independent criterion and if represented in the generalized shear-pressure space, is a straight line. This is shown in Figure 1.

The success of the general plasticity theory motivated researchers to expand the scope of the theory and to develop plasticity models to capture the behavior of frictional and cementitious type materials such as concrete. There are numerous multi-parameter plasticity-based models published in the literature and most nonlinear finite element codes incorporate a number of these plasticity models. The simplest one is the Drucker-Prager formulation that is also shown in Figure 1 to be contrasted with the von Mises surface. In the Drucker-Prager model, the material is pressure-dependent where the uniaxial compressive strength is several times greater than the uniaxial tensile strength and where the material exhibits enhanced strength and ductility under the increasing pressure. This is one well known characteristic of the behavior of concrete.

Non-linearity in concrete arises from two distinct meso-structural changes. One is the development of cracks and microcracks under zero or low confining pressures, and the other is development of plastic flow due to void closure and slips planes in the aggregate field in the presence of large confining pressures (Ortiz 1985). The objective of this paper is to that by considering the stability definitions for thermodynamics potential functions, the Drucker's plasticity postulate can be obtained as a sub-set of an over-all stability statements. The same can be expanded to establish a softening stability postulate for damage processes due to microcracking as will be shown in the sequel.

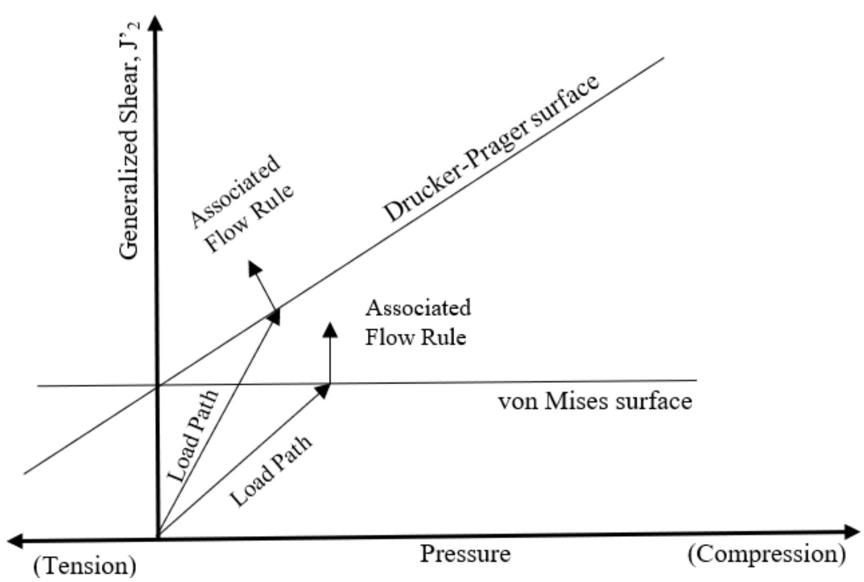

Figure 1. Representation of von Mises and Ducker-Prager surfaces in the generalized shear-pressure space.

Using the internal variable theory of thermodynamics and utilizing the Gibbs Free Energy (GFE), G(σ, *q*), the dissipation inequality can be shown (Eq. 2) to yield (Lubliner 1972).

$$\frac{\partial G(\sigma,q)}{\partial q}\dot{q} = \frac{\partial G(\sigma,q)}{\partial \gamma}\dot{\gamma} + \frac{\partial G(\sigma,q)}{\partial k}\dot{k} \geq 0 \tag{2}$$

where, the internal inelasticity parameter q is given by $q = k + \gamma$ in which k is identified as the cumulative damage parameter and γ represents the plasticity parameter. The second law of the thermodynamics also leads to the conclusions that GFE is a potential function for the total strain tensor, i.e. $\varepsilon = \partial G(\sigma, q)/\partial\sigma$ so that the general form of the GFE can be stated as shown in Eq. (3) (Yazdani and Karnawat 1996):

$$G(\sigma, q) = \frac{1}{2}\sigma: C(k): \sigma + \varepsilon^p: \sigma - A^i(q) \tag{3}$$

where, C(k) denotes the current compliance of the material and is assumed to change with the initiation and accumulation of damage, and where $A^i(q)$ is the inelastic component of the Helmholtz Free Energy (HFE) that arises as a constant of integration. Two functions Φ and Ω are then defined in Eq. (4) such that

$$\Phi(\sigma, \gamma) = \frac{\partial G(\sigma,q)}{\partial\gamma} \ \& \ \Omega(\sigma, k) = \frac{\partial G(\sigma,q)}{\partial k} \tag{4}$$

It is further assumed that $\dot{\gamma}$ and $\dot{k}$ satisfy the irreversibility conditions, namely $\dot{\gamma} \geq 0$ & $\dot{k} \geq 0$, and that no coupling between damage and plasticity takes place. This is referred to as an un-coupled theory. To progress further, the rate of the total strain tensor is decomposed into plastic and damage strain components as shown in Eq. (5):

$$\dot{\epsilon} = \dot{\epsilon}^p(\gamma) + \dot{\epsilon}^D(k) \tag{5}$$

where the plastic strain tensor is denoted by $\dot{\boldsymbol{\epsilon}}^{\boldsymbol{p}}$ as before and the damage strain tensor is denoted by $\dot{\boldsymbol{\epsilon}}^{\boldsymbol{D}}$. The additive decompositions of tensors, such as shown above, is allowed for small deformations as assumed here for concrete.

2 FLOW RULES FOR PLASTICITY

With the assumption that concrete is a plastically stable material and that the softening behavior is only due to the process of damage (Ortiz 1985), the following form of the yield function for plasticity is postulated in Eq. (6).

$$F(\sigma, \gamma) = F^*(\sigma) - \tau(\gamma) \tag{6}$$

where $F^*(\sigma)$ is a homogenous function in σ of the first degree and where $\tau(\gamma)$ defines the hardening rule. Then, the plastic strain rate is given as in Eq. (7).

$$\dot{\epsilon}^p(\gamma) = \dot{\gamma}\frac{\partial F(\sigma,\gamma)}{\partial\sigma} = \dot{\gamma}\frac{\partial F^*(\sigma)}{\partial\sigma} \tag{7}$$

and utilizing Eqs (3) and (4), it yields to Eq. (8).

$$\Phi(\sigma, \gamma) = \sigma: \frac{\partial\epsilon^p(\gamma)}{\partial\gamma} - \frac{\partial A^p(\gamma)}{\partial\gamma} = \sigma: \frac{\partial F^*(\sigma)}{\partial\sigma} - \tau(\gamma) = F^*(\sigma) - \tau(\gamma) \tag{8}$$

where, A^p is the plastic component of the $A^i(q)$ and where use has been made of the Euler theorem for homogenous functions and $\tau(\gamma)$ is chosen to be equal to $\frac{\partial A^p(\gamma)}{\partial\gamma}$ so that function Φ and F are the same.

3 FOLLOW RULES FOR DAMAGE

For small deformations as is appropriate for brittle materials, the total flexibility tensor can be divided into the initial undamaged flexibility, C^0, and added flexibility, $C^c(k)$, due to microcracking; that is $C(k) = C^0 + C^c(k)$. The changes in the damage strain tensor are then related to the changes in the added flexibility tensor $\dot{C}^C$ as shown in Eq. (9).

$$\dot{\epsilon}^D(k) = \dot{C}^C(k): \sigma \tag{9}$$

so that $\dot{\epsilon}^D(k)$ can be obtained by identifying an evolution equation for $\dot{C}^C(k)$. This can be accomplished if a fourth-order response tensor $R(\sigma)$ is defined in Eq. (10) such that

$$\dot{C}^C(k) = \dot{k}R(\sigma) \tag{10}$$

Different form of response tensors would identify different damage model. For example, an isotropic damage formulation can be obtained if one identifies $R(\sigma)$ to be proportional to the general isotropic compliance tensor of elasticity (Yazdani *et al*. 2019). Following the general damage formulation of Ortiz (1985) and using Eq (4), the damage potential function, $\Omega(\sigma, k)$ is identified in Eq. (11) as

$$\Omega(\sigma, k) = \frac{1}{2}\sigma: \frac{\partial C^C(k)}{\partial k}: \sigma - \frac{\partial A^D(k)}{\partial k} \tag{11}$$

where, the function A^D reflects the damage component of $A^i(q)$ associated with the surface energy of cracks formation. A general from of the damage surface, $\Psi(\sigma,k)$, can then be formulated by defining a softening function f(k) such that $\Psi(\sigma, k) = \Omega(\sigma, k) - f^2(k) = 0$ subject to the condition that $f(\partial f/\partial k) \leq 0$. Other possible damage formulations for concrete can be found in the literature (Vorobiev *et al*. 2018, Li and Wu 2018, Peng and Meyer 2000).

With the irreversibility assumptions on γ and k and in the absence of any internal constraints, it follows that the criteria for loading and unloading can formally be stated in the standard Kuhn-Tucker forms for both the plasticity and damage processes as shown in Eq. (12) and Eq. (13).

$$\Phi \leq 0,\ \dot{\gamma} \geq 0,\ \dot{\gamma}\Phi = 0 \tag{12}$$

$$\Psi \leq 0,\ \dot{k} \geq 0,\ \dot{k}\Psi = 0 \tag{13}$$

4 STABILITY CONSIDERATION

In order to investigate stability, certain definitions and assumptions must be introduced (Simo and Hughes 1988, William *et al*. 1985) as shown through Eq. (14) – Eq. (30). Considering Gibbs Free Energy function, G, the two subset parameters q_i and q_j are mutually hardening if

$$\dot{q}_i \frac{\partial^2 G}{\partial q_i \partial q_j} \dot{q}_j \leq 0 \tag{14}$$

and are mutually softening (Eq. (15)) if

$$\dot{q}_i \frac{\partial^2 G}{\partial q_i \partial q_j} \dot{q}_j \geq 0 \tag{15}$$

A subset parameter q_α is stable, if there is hardening with respect to itself that is,

$$\dot{q}_\alpha \frac{\partial^2 G}{\partial q_\alpha \partial q_\alpha} \dot{q}_\alpha \leq 0 \tag{16}$$

and is unstable if there is softening with respect to itself, that is,

$$\dot{q}_\alpha \frac{\partial^2 G}{\partial q_\alpha \partial q_\alpha} \dot{q}_\alpha \geq 0 \tag{17}$$

Concrete is assumed to be plastically stable, that is,

$$\dot{\gamma} \frac{\partial^2 G}{\partial \gamma \partial \gamma} \dot{\gamma} \leq 0 \tag{18}$$

and concrete is assumed to be unstable under damage, that is,

$$\dot{k}\frac{\partial^2 G}{\partial k \partial k}\dot{k} \geq 0 \tag{19}$$

With these definitions and assumptions, the consistency condition for rate independent plasticity will be examined first. Considering Φ:

$$\dot{\Phi} = \frac{\partial \Phi(\sigma,\gamma)}{\partial \sigma} : \dot{\sigma} + \frac{\partial \Phi(\sigma,\gamma)}{\partial \gamma}\dot{\gamma} = \frac{\partial F^*(\sigma)}{\partial \sigma} : \dot{\sigma} - \frac{\partial \tau(\gamma)}{\partial \gamma}\dot{\gamma} = 0 \tag{20}$$

and, denoting $\frac{\partial \tau(\gamma)}{\partial \gamma} = H^p(\gamma)$ as the plastic modulus, Eq. (20) becomes

$$\frac{\partial F^*(\sigma)}{\partial \sigma} : \dot{\sigma} = H^p(\gamma)\dot{\gamma} \tag{21}$$

From Eq. (18) and with the irreversibility assumption on $\dot{\gamma}$, it follows that

$$\frac{\partial^2 G}{\partial \gamma \partial \gamma} = -\frac{\partial \tau(\gamma)}{\partial \gamma} = -H^p(\gamma) \leq 0 \tag{22}$$

which implies that

$$H^p(\gamma) \geq 0 \tag{23}$$

that is, the plastic modulus is nonnegative. After multiplying the terms in Eq. (23) with $\dot{\gamma}$, it follows that

$$\dot{\gamma}\frac{\partial F^*(\sigma)}{\partial \sigma} : \dot{\sigma} = \dot{\gamma}H^p(\gamma)\dot{\gamma} \geq 0 \tag{24}$$

thus

$$\underline{\dot{\epsilon}^p : \dot{\sigma} \geq 0} \tag{25}$$

which is the same as Drucker's stability postulate. With the same approach, one can now consider the consistency conditions for the damage surface, as:

$$\dot{\psi} = \frac{\partial \psi(\sigma,k)}{\partial \sigma} : \dot{\sigma} + \frac{\partial \psi(\sigma,k)}{\partial k}\dot{k} = 0 \tag{26}$$

or

$$\left(\frac{\partial C^C(k)}{\partial k} : \sigma\right) : \dot{\sigma} + \left[\left(\frac{1}{2}\sigma : \frac{\partial^2 C^C(k)}{\partial k^2} : \sigma\right) - \frac{\partial^2 A^D(k)}{\partial k^2} - 2f(\partial f/\partial k)\right]\dot{k} = 0 \tag{27}$$

From Eq. (19), and that $\dot{k} \geq 0$, it follows that

$$\frac{1}{2}\sigma : \frac{\partial^2 C^C(k)}{\partial k^2} : \sigma - \frac{\partial^2 A^D(k)}{\partial k^2} - 2f(\partial f/\partial k) \geq 0 \tag{28}$$

which, when compared to Eq. (27), yields that

$$\left(\frac{\partial C^C(k)}{\partial k} : \sigma\right) : \dot{\sigma} \leq 0 \tag{29}$$

The multiplication of terms in Eq. (29) with $\dot{k} \geq 0$ leads to

$$\left(\dot{k}\frac{\partial C^C(k)}{\partial k} : \sigma\right) : \dot{\sigma} = \underline{\dot{\epsilon}^D : \dot{\sigma} \leq} 0 \tag{30}$$

which, represents the stability statement for the damage-softening processes. All admissible damage processes must satisfy the inequality of Eq. (30) and damage models must be formulated such that Eq. (30) is not violated.

5 CONCLUSIONS

In this paper, the stability conditions of inelastic deformations in concrete are arrived at by utilizing dissipation inequalities of the internal variable theory of thermodynamics. It is shown that by a set of stability statements of thermodynamic potentials and assuming associate flow rules, the Drucker's stability postulate can be reached showing the positive rate of work done of the plastic strain and the Cauchy stress changes. Similarly, the approach can be extended to establish the softening character of damage due to microcracking in concrete. Eqs (25) and (30) represent the two stability conditions for plasticity and for damage, respectively. Throughout this paper, it is assumed that elasto-plastic-damage processes are un-coupled. Consideration of coupled theories were beyond the scope of this work.

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