BUCKLING STRENGTH OF COLUMNS IN LOW-RISE MULTI-SPAN FRAMES EVALUATED BY THE STABILITY INDEX

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The objective of this study was to investigate the validity of the effective length factors for low-rise multi-span frames obtained by the stability index (SI) method, and SI is defined by the ratio of $P\Delta$ moment to story moment. The SI method calculates the buckling load based on the condition that the story with the maximum stability index is the first to lose its stiffness compared with the other stories at the time of buckling. The analytical parameters used in the present study were the stiffness ratios of beams and columns. The effective length factors obtained by the SI method were compared with the exact effective length factors provided by the buckling slope-deflection method. The story displacement $\Delta$ of the $P\Delta$ moments was calculated by the application of horizontal forces at the top of the column of each story. This paper outlines a valid process to evaluate the effective length factors using the SI method and demonstrates the impact of the horizontal force distribution and $P\delta$ moments on the accuracy of the SI method in low-rise, multi-span frames.

Keywords: Frame stability, Buckling slope-deflection method, SI method, $P\Delta$ moment, $P\delta$ moment.

1 INTRODUCTION

For stability design, the direct analysis method is presented in AISC Specification (AISC 2016). However, the effective length factor of columns has to be calculated in design of columns for allowable stress design of steel structures, and the effective length factor is usually evaluated by Architectural Institute of Japan Standards in Japan (AIJ 2019). As a result, the stability index (SI) method, which provides a simple way to calculate the effective length factor, was recently reported (Kido and Tsuda 2020). In addition, the recently proposed formula for elastic buckling load of columns in nonuniform frames has been shown to evaluate the buckling load accurately (Takada 2021). However, the specific method that is appropriate for multistory, multi-span frames has yet to be presented. The SI method is a simple way to evaluate the effective length factor of columns in a frame (Kido and Tsuda 2020), yet the $P\delta$ effect due to axial forces and bending deformation of columns is not considered. Therefore, the accuracy of the SI method is bound to be compromised in single-story frames with fixed bases or when the end restraints of column members increase.

In this study, the effective length factor of column members in low-rise multi-span frames with bottom beams, pinned bases, and fixed bases is calculated using the SI method and compared with the correct values obtained by the buckling slope-deflection method. Furthermore, the present study demonstrates to modify the effective length factor obtained by the SI method using coefficients that account for the effect of $P\delta$ moment (Takada 2021).
2 ANALYSIS

2.1 Problem Setting

In this paper, the buckling load of $m$-story, $n$-span frames is calculated. The analytical model is shown in Figure 1. All columns ($I$: moment of inertia of area, $h$: length) and beams have the same cross-section and length ($k_b$: stiffness ratio of bottom beams, $k_s$: stiffness ratio of beams above the second story, $l$: length). For compressive load, the side columns were assumed to be subjected to a vertical force $P$ at the column head at each story, while the middle columns were assumed to be subjected to a vertical force $2P$. The calculation was carried out by the SI method (Kido and Tsuda 2020).

\[
\sum_{i=1}^{n} P_i \cdot A = \sum_{i=1}^{n} \left( \xi_i \cdot (m-i+1)P \right) \Delta / Q \cdot h
\]

Figure 1. Analytical model used in this study.

2.2 Calculation of the Buckling Load and Effective Length Factor Using the SI Method

To calculate the buckling load using the SI method, the following steps were followed:

(i) Apply only the horizontal force $H_i$ to each story $i$ of the frame shown in Figure 1 and calculate the story shear force $Q_i$ and the story displacement $\Delta_i$ for each story $i$ ($i = 1$--$m$ for $m$-story frames).

(ii) The SI is calculated by using Eq. (1). $P_{ij}$ in the middle of Eq. (1) is the axial load of the $j$-th column in story $i$. $P_{ij}$ can be expressed as the numerator of the right side of Eq. (1) using the compressive load ratio $\xi_{ij}$ for the $i$-story $j$-th column ($j = 1$--$n+1$ for $n$-span frames). In the analytical model shown in Figure 1, $\xi_{ij}$ is one for the side columns and two for the middle columns.

\[
SI_i = \sum_{j=1}^{n+1} P_{ij} \cdot A = \sum_{j=1}^{n+1} \left( \xi_{ij} \cdot (m-i+1)P \right) \Delta / Q \cdot h
\]

(iii) The SI of each story $i$ is obtained using Eq. (1), and the story $i_{cr}$ with the largest SI is considered as the buckling associated story (hereinafter referred to as BAS).

(iv) The buckling equation is defined when the SI of BAS is equal to one, and Eq. (2) is obtained. The axial force $P$ that satisfies Eq. (2) is denoted as $\tilde{P}$ and the buckling load $P_{kl}$ of any $k$-story $l$-th column can be expressed as in Eq. (3) by using $\tilde{P}$.

\[
\sum_{j=1}^{n+1} \left( \xi_{cr,j} \cdot (m-i_{cr}+1)\tilde{P} \right) \Delta_{cr} / Q_{cr} \cdot h = 1
\]

\[
P_k = \xi_{kl} \cdot (m-k+1) \cdot \tilde{P}
\]
(v) The effective length factors $K_{kl}$ of any $k$-story $l$-th column are given by Eq. (4) using the Euler’s load of the column $P_{el}$. In this paper, the effective length factors of the first-story middle column $K_{SI}$ are shown as analytical results.

$$K_{kl} = \sqrt[4]{\frac{P_{el}}{P_{SI}}}$$ (4)

The analytical parameters are as follows: 1) number of spans $n = 3, 5, 10$; 2) number of story $m = 2, 3, \text{and } 4$; 3) beam stiffness ratio $k_r = 0–5$; 4) type of frames: frame with bottom beam ($k_t = k_b$), fixed bases, pinned bases, and 5) horizontal force distribution Case I: the distribution of axial force and story shear in each story are set to be the same, Case II: The distribution of story shear is set to be the ratio of the $A_i$ distribution, where $A_i$ is a coefficient used in seismic design in Japan that indicates the distribution of seismic story shear force along the height direction.

3 RESULTS AND DISCUSSION

3.1 Relationship between Effective Length Factor and the Stiffness Ratio of Beam

Figure 2 shows the relationship between the effective length factors and the stiffness ratio of beams of the first-story middle column of a three-story, five-span frame. The solid black line represents the correct value of $K_{exact}$ obtained by the buckling slope-deflection method, and the solid blue and red lines reflect the values obtained by the SI method for Case I and Case II horizontal force distributions, respectively. Also, BAS was the first story for frames with bottom beams and pinned bases regardless of the value of $k_t$. In contrast, for frames with fixed bases, BAS changed from the second to the first story at $k_t = 0.924$ and 0.749 in Case I and Case II, respectively.

![Figure 2. Relationship between the effective length factors and stiffness ratio of the beam.](image)

3.2 Relationship between Error and Various Parameters

Figure 3 demonstrates the relationship between the error $E$ and the stiffness ratio of beam $k_t$ for a three-story, five-span frame. Figure 3(a), (b), and (c) show the frame with bottom beams, fixed bases, and pinned bases, respectively. Errors are shown as percentages of the relative difference between the effective length factors $K_{SI}$ obtained by the SI method and the correct values $K_{exact}$. According to Figure 3, there is a range of $k_t$ values ($E > 0$) for which the buckling load obtained by the SI method is smaller than the correct value, regardless of the type of frame and horizontal force distribution.

In contrast, for the range of $k_t$ values for which the buckling load obtained by the SI method is larger than the correct value ($E < 0$), the error increases as the stiffness ratio of the beam
increases. The error trends for the frames with bottom beams and pinned bases (Figure 3(a) and (c), respectively) are similar to each other, and the difference between the error curves of Case I and Case II decreases as the stiffness ratio of the beam increases. Therefore, it can be assumed that the influence of the difference in horizontal force distribution on the accuracy of the SI method decreases as the stiffness ratio of the beam increases. When it comes to frames with fixed bases (Figure 3(b)), the error curves in both cases almost overlap with each other for stiffness ratios for which the BAS is the second story, indicating that the influence of the horizontal force distribution on the accuracy of the SI method is negligible. However, the change in error becomes moderate when the BAS switches from the second story to the first.

![Graph 3](image3.png)

Figure 3. Relationship between error $E(\%)$ and stiffness ratio of the beam (3-story 5-span frame).

![Graph 4](image4.png)

Figure 4. Relationship between error $E(\%)$ and stiffness ratio of the beam.

Figure 4 shows the effect of the type of frame, the number of stories, and the horizontal force distribution on accuracy of the SI method for 2-story, 3-story, and 4-story frames (Figure 4(a), (b), and (c), respectively), where the stiffness ratio of the beam on the horizontal axis is 0.5, 1.0, 1.5, and 2.0, respectively. The blue and red plots indicate the horizontal force distributions of Case I and Case II, respectively. Furthermore, the circle, square, and triangle plots indicate frames with bottom beams, fixed bases, and pinned bases, respectively. For each beam stiffness ratios, the errors for three-span, five-span, and ten-span frame are plotted from left to right. According to this figure, the overall evaluation can be performed within a 6% error regardless of the type of frame, horizontal force distribution, number of spans, and number of stories, provided that the stiffness ratio of the beam is within 0.5–2.0. The error variability decreases with increasing number of stories, suggesting that the type of frame and horizontal force distribution have a smaller effect on the accuracy of the SI method.

4 MODIFICATION OF THE EFFECTIVE LENGTH FACTOR

The previous section demonstrated that the error $E$ tends to increase as the stiffness ratio of the beam increases when the buckling load obtained by the SI method is larger than the correct value.
This is because the bending deformation of columns becomes larger as degree of the restraints of beams become greater, and the effect of $P\delta$ moment on the error become larger. However, the SI method intends to evaluate the effective length factors in a simple way and does not consider the $P\delta$ effect generated in the column. A formula for evaluating the elastic buckling loads of columns in nonuniform frames, including the coefficient $D_j$ (Eq. (5)) that considers the effect of axial shortening, i.e., bending deformation, was previously proposed (Takada 2021).

$$D_j = \frac{6\left((k_{rT} + 4)(k_{rB} + 4) - 6\right)^2 - k_{rT}k_{rB}(k_{rT} + k_{rB} + 10) + 20}{(k_{rT} + 4)(k_{rB} + 4) - 4}$$

(5)

where $k_{rT}$ and $k_{rB}$ are the rotational stiffness of the column ends nondimensionalized by $EI_c / h$, corresponding to the top and bottom ends of the column, respectively. ($EI_c$: flexural rigidity of the column). The rotational stiffness considers the end restraints provided by the surrounding members attached to the column. The coefficient $D_j$ is 1.0 (minimum value) when both ends of the column are pinned and 1.2 (maximum value) when both ends are fixed.

In this section, attempt to modify the effective length factor using the SI method by using the coefficient $D_j$, which can consider the effect of $P\delta$ effect. Also, in AISC Specification (AISC 2016), a constant value $R_M (= 0.85)$ is shown as a coefficient to account for the $P\delta$ effect, which is a value obtained when both ends of the column are fixed, and corresponds to a $D_j = 1.2$.

### 4.1 Correcting the Effective Length Factor Using the SI Method

After proceeding to step (iii) in section 2.2 and obtaining the BAS, the buckling load of the column in the BAS is modified as follows. Since the coefficient $D_j$ is greater than or equal to 1, it can be interpreted as an amplification factor for the story displacement $\Delta_i$ in Eq. (1). Furthermore, Eq. (2) can be modified as Eq. (6) by multiplying $D_j$ with $D_j$. Hence, $\tilde{P}$ in Eq. (6) is the modified buckling load. The modified effective length factors $K_{SI}^{mod}$ can be obtained by the same procedure described in steps (iv) and (v) in section 2.2. The relationship between $K_{SI}^{mod}$ and $K_{SI}$ before the modification is expressed by Eq. (7).

$$\frac{\sum_{j=1}^{nS1} \left[ \xi_{\nu j} \cdot (m - i_r + 1) \tilde{P} \right] \cdot (D_j \cdot A_{\nu j})}{Q_{\nu j} \cdot h} = 1$$

(6)

$$K_{SI}^{mod} = \sqrt{\frac{\sum_{j=1}^{nS1} \xi_{j} \cdot D_j}{\sum_{j=1}^{nS1} \xi_{j} \cdot K_{SI}}}$$

(7)

To simplify the analysis, it is assumed that only beams attached to the column restrain the both ends of the column, and all flexural rigidity of beams are assumed to contribute to the restraint of the column ends. All beam deformations are assumed to be inversely symmetric bending deformations.

To correct the effective length factor using the coefficient $R_M$, simply replace the part of $D_j$ in Eq. (6) and Eq. (7) with $1 / R_M$. Consequently, the effective length factors obtained by the SI method may be nearly 14% larger than the effective length factors obtained by the buckling slope-deflection method.
4.2 Relationship between Error and Stiffness Ratio of Beams

Figure 5 shows the errors of the effective length factors modified by the coefficient $D_i$ relative to the correct values.

According to Figure 5, the modified effective length factors are greater than the correct values ($E > 0$) in most cases and can be evaluated within approximately 5% in many cases compared to the error distribution shown in Figure 4. This can be explained by the fact that the coefficient $D_i$ considers the degree of restraint at the top and bottom ends of the columns, as shown in Eq. (5). In addition, for the stiffness ratios greater than 1.0, the evaluation can be performed accurately with an error of approximately 5% independently of the type of frame, the number of spans, the number of stories, or the horizontal force distribution. However, in cases where the stiffness ratio is 0.5 and the horizontal force distribution is the one described in Case II, the effective length factors may be overestimated against the correct value. Therefore, it is necessary to carefully consider the influence of the horizontal force distribution for small stiffness ratios, as mentioned in (Takada 2021), however this is an issue for future research studies.

![Graph showing the relationship between error and stiffness ratio of beams for different frames and cases.]](image)

Figure 5. Relationship between error and stiffness ratio of the beam (modified effective length factor).

5 CONCLUSIONS

The conclusions derived from this study are as follows:

(i) Within the range of this analysis, the SI method could evaluate the stiffness ratio of beams in the range of 0.5–2.0 with an error of approximately 6%.

(ii) The effective length factor modified by $D_i$ can be evaluated within approximately 5% accuracy ($E > 0$) for the stiffness ratios greater than 1.0 independently of the type of frame, the number of spans, the number of stories, or the horizontal force distribution.

References


